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STUDENT VERSION
PENDULUM MODELING

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STATEMENT

Massless Rod Pendulum

Let us consider a massless rod (say, a plastic straw) of length L m which can freely rotate about a fixed point O, and a mass m attached to the end of the rod as sen in Figure 1. As the rod rotates there is a centripetal force exerted on the mass in the direction perpendicular to the circular path. However, this force is counteracted by the tension force in the rod itself and so the only force we need to contend with acting on the mass is that due to gravity, namely $-m \cdot g$, where g is the acceleration due to gravity. Our convention is that both negative force and velocity are acting downward.

a) Ideal Pendulum

We shall presume we have purchased this pendulum apparatus from the "Ideal Pendulum Company" and that there is no resistance to the motion, nor friction in the mechanism at O. We note that even though this is untrue we shall produce a model and do a reality check on it to see if the final motion really reflects our assumptions.

i) Build a differential equation model for this pendulum motion by constructing a Free Body Diagram for the mass m kg at the end of the rod and considering the External Forces tangential to the circular path of the pendulum in Newton's Second Law of Motion formulation. Hint: $\theta \cdot L$ is the length of the circular path the mass m at the end of the pendulum has traveled when the arc of θ radians has been swept in the pendulum motion where $\theta(t)$ is the angle of the pendulum with the vertical line at time t s. Then $m \cdot \theta''(t) \cdot L$ is the $m \cdot a$ term we need to begin our model using Newton's Second Law of Motion. We can set this force equal to the sum of the External Forces acting on the mass.

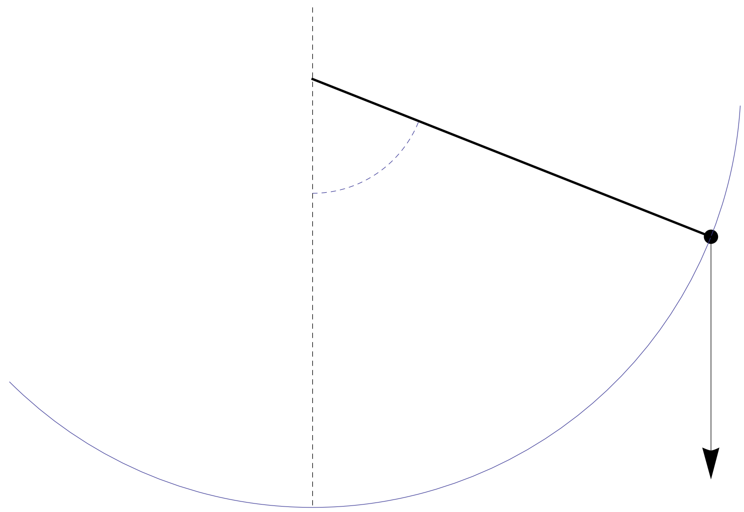

Figure 1. Simple pendulum diagram, depicting rotation point O, angle of rotation θ, pendulum rod of length L, mass m at end of rod and weight - and driving force for action - of mass downward.

ii) Upon formulation of the model use $L = 0.3$ m and $m = 0.2$ kg, with initial conditions $\theta(0) = \frac{\pi}{3}$ and $\theta'(0) = 0$, and a numerical solution routine for differential equaitons over a reasonable time interval $[0, 10]$ s, say, to determine $\theta(t)$.

iii) Then plot (a) $\theta(t)$ vs. t (see Figure 2 for what you should obtain) and (b) $\theta'(t)$ vs. $\theta(t)$ - usinga parametric plotting routine. The latter is called a *phase portrait* (see Figure 3 for what you should obtain). Explain what your plots show and why these plots are reasonable in light of our assumptions.

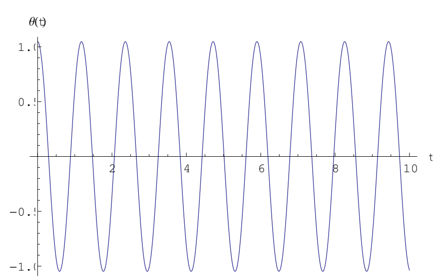

Figure 2. Plot of characteristic solution to simple pendulum model.

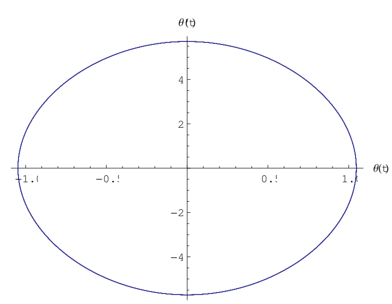

Figure 3. Plot of phase portrait for characteristic solution to simple pendulum model.

iv) Describe what is unrealistic about your pendulum model based on the graphics of Figure 2 and Figure 3.

b) More Realistic Pendulum

We ought to consider some resistance term in our model, perhaps the familiar resistance term proportional to the velocity of our mass, either due to friction at O and/or resistance due to the medium and the mass m.

i) Introduce a resistance term to your accounting for part (a). Recall we are summing our External Forces tangential to the circular path of the pendulum and the linear distance traversed along the circular path as we go through θ radians is $L\theta$. Thus, a good resistance term would be $c \cdot L \cdot \theta'(t)$. Use this notion to modify your differential equation from (a) to include resistance. Use $c = 0.1$ N/(m/s).

ii) Upon formulation of the model use $L = 0.3$ m, $m = 0.2$ kg, and, $c = 0.1$ N/(m/s) with initial conditions $\theta(0) = \frac{\pi}{3}$ and $\theta'(0) = 0$ in a numerical solution routine for differential equaitons over a reasonable time interval $[0, 20]$ s, say, to determine $\theta(t)$.

iii) Then plot (a) $\theta(t)$ vs. t (see Figure 4 for what you should obtain) and (b) $\theta'(t)$ vs. $\theta(t)$ using a parametric plot routine. See Figure 5 for what you should obtain. Explain what your plots show and why these plots are reasonable in light of our assumptions.

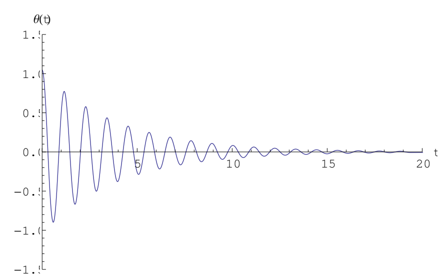

Figure 4. Plot of characteristic solution to simple damped pendulum model.

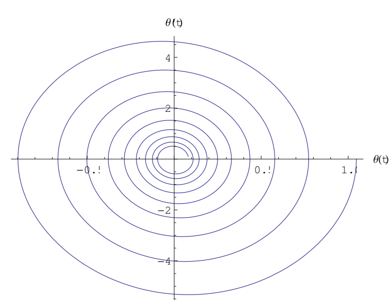

Figure 5. Plot of phase portrait for characteristic solution to simple damped pendulum model.

iv) Now use the same model parameters from (ii), $L = 0.3$ m, $m = 0.2$ kg, and, $c = 0.1$ N/(m/s), but with new initial conditions $\theta(0) = \frac{\pi}{3}$ and $\theta'(0) = 20$. Notice that we have given our pendulum a giant initial velocity and thus things might (ought to) change in our imagery. Then plot (a) $\theta(t)$ vs. t (see Figure 6 for what you should obtain) and (b) $\theta'(t)$ vs. $\theta(t)$ using a parametric plot routine. See Figure 7 for what you should obtain. Explain what your plots show and why these plots are reasonable in light of our assumptions. Be particularly attuned to the effects of a large initial velocity.

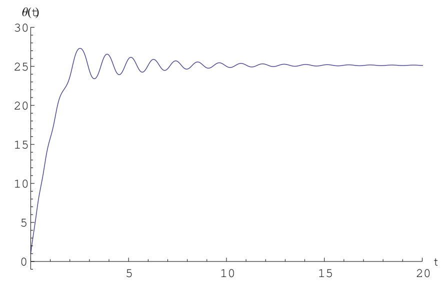

Figure 6. Plot of characteristic solution to simple damped pendulum model with high initial velocity.

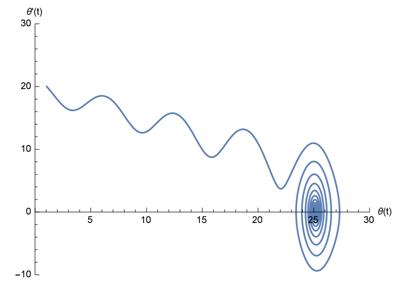

Figure 7. Plot of phase portrait for characteristic solution to simple damped pendulum model with high initial velocity.

Linear Approximation of Massless Pendulum

If we have built a differential equation model for a massless rod of length L m pendulum with no resistance $c = 0$ N/(m/s) and mass m kg at the end of the rod it should look like (1):

$$m \cdot L \cdot \theta''(t) = -m \cdot g \cdot \sin(\theta(t))\,, \tag{1}$$

or in more familiar manner,

$$m \cdot L \cdot \theta''(t) + m \cdot g \cdot \sin(\theta(t)) = 0\,. \tag{2}$$

For small angle swings, $\theta(t) << 1$ we might replace the nonlinear term $-m \cdot g \cdot \sin(\theta(t))$ in (1) with just $-m \cdot g \cdot \theta(t)$. This is because $\theta \approx \sin(\theta)$ for small θ. So in this case we are back to a familiar differential equation (3):

$$m \cdot L \cdot \theta''(t) + m \cdot g \cdot \theta(t) = 0\,. \tag{3}$$

Equation (3) with initial conditions $\theta'(0) = 0$ and $\theta(0) = \theta_0$ has the following solution (can you show this?):

$$\theta(t) = \cos\left(\sqrt{\frac{g}{L}}t\right)\,. \tag{4}$$

We see an advantage in using the approximation $\theta \approx \sin(\theta)$, for we get a reasonable estimate of the frequency $\left(\frac{1}{2\pi}\sqrt{\frac{g}{L}}\right)$ in Hz and period $\left(2\pi\sqrt{\frac{L}{g}}\right)$ in s of the pendulum's oscillation. We can check these with a stop watch, say, by taking the time for 10 complete cycle swings and dividing by 10 to obtain the period or time for one cycle. We note that this period is independent of the mass m hung at the end of the massless rod of the pendulum and of the initial angular position, $\theta(0) = \theta_0$. Of course, we know this model is not real, for it never decays, but for small angles and very little friction or resistance situation this is a reasonable approximate answer with useful information revealed because of it.

i) Solve both differential equation (2) without the approximation $\theta \approx \sin(\theta)$ for small θ and differential equation (3) with the approximation $\theta \approx \sin(\theta)$ for small θ using the parameters $m = 0.4$ kg, $L = 0.3$ m, $\theta_0 = 0.1$and $g = 9.8$, m/s^2.

ii) Compare your solutions graphically by plotting $\theta(t)$ vs. t.

ii) Also compare their periods. How will we determine the period in the case of differential equation (2)?

iii) Address both (i) and (ii) with the same parameters but change θ_0 to $\theta_0 = 1.4$, i.e. a non-small angle.

Physical Pendulum

Let us consider a physical (real!) pendulum. That is, let us acknowledge that our rod of length L m which can freely rotate about a pivot point O has a distributed mass of its own as well as a mass m attached to the end of the rod as sen in Figure 1. As the rod rotates there is a centripetal force exerted on the mass in the direction perpendicular to the circular path. However, this force is counteracted by the tension force in the rod itself and so the only force we need to contend with acting on the mass is that due to gravity, namely $-m \cdot g$, where g is the acceleration due to gravity. Our convention is that both negative force and velocity are acting downward.

We will assume that our rod has a varying mass density $\rho(x)$ along it length from the pivot point O $(x = 0)$ to the tip $(x = L)$ and further we will permit masses $\{m_i \,|\, i = 1, 2, \ldots, n\}$ to be hung on the rod, at distances $\{x_i \,|\, i = 1, 2, \ldots, n\}$, respectively, from the pivot point O.

So the mass M of our physical pendulum will be, M, as computed in (5)

$$M = \int_0^L \rho(x)\, dx + \sum_{i=1}^{n} m_i\,, \tag{5}$$

the center of mass of the rod will be $\bar{x}$ as computed in (6)

$$\bar{x} = \frac{\int_0^L x\,\rho(x)\, dx + \sum_{i=1}^{n} x_i \cdot m_i}{M}\,, \tag{6}$$

and our moment of inertia about the pivot point O will be I_O as computed in (7)

$$I_O = \int_0^L x^2\,\rho(x)\, dx + \sum_{i=1}^{n} x_i^2 \cdot m_i\,. \tag{7}$$

Now if we took a rod of uniform mass density (not hard to realize), $\rho(x) = \rho$ we have the appropriate center of mass, $\bar{x}$, (8) and moment of inertia about O, I_O, (9):

$$\bar{x} = \frac{\frac{\rho \cdot L^2}{2} + \sum_{i=1}^{n} x_i \cdot m_i}{M}\,, \tag{8}$$

$$I_O = \frac{\rho L^3}{3} + \sum_{i=1}^{n} x_i^2 \cdot m_i\,. \tag{9}$$

Finally, we seek the radius at which we might concentrate all this mass so that the physical pendulum and the theoretical (massless as above) pendulum behave the same. This distance is called the *center of oscillation*, L_O, and is computed in (10),

$$L_O = \frac{I_O}{M \cdot \bar{x}}\,. \tag{10}$$

i) Suppose we have a physical spring of length $L = 0.3$ m with uniform mass density $\rho = 0.05$ kg/m and several masses at distinct locations $m_1 = 0.04$ kg at $x_1 = 0.1$ m, $m_2 = 0.3$ kg at $x_2 = 0.15$ m, with our original mass $m = m_3 = 0.2$ kg at $x_3 = 0.3$ m. We presume the same resistance parameter $c = 0.1$ N/(m/s) and initial conditions $\theta(0) = \frac{\pi}{3}$ and $\theta'(0) = 0$ as with our massless pendulum.

   Build the differential equation for this physical pendulum with resistance.

ii) Solve this resulting differential equation you built and compare this solution with that of the massless rod with resistance under the same conditions you solved above. Render up and compare the plots of $\theta(t)$ vs. t and the parametric plots of $\theta'(t)$ vs $\theta(t)$.

iii) Given observational data on a physical pendulum so we have all but c, how would you estimate c?