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STUDENT VERSION

World Population

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STATEMENT

We seek to derive a general model for world population growth using a differential equation.

Let us denote $N(t)$ the size of the world (human) population at time t in years. Next, let $r(t)$ represent the growth rate of this population (percentage increase of population) at time$t$. From this we can see the change in the world population over a period of time Δt.

$$N(t + \Delta t) = N(t) + r(t) \cdot N(t) \cdot \Delta t\,. \tag{1}$$

Question 1. Defend, in words, this model in (1).

We adjust this relationship to form

$$\frac{N(t + \Delta t) - N(t)}{\Delta t} = r(t) \cdot N(t)\,. \tag{2}$$

and by the limit transition, $\Delta t \to 0$, we get the resulting differential equation

$$\frac{dN}{dt} = r(t) \cdot N(t)\,. \tag{3}$$

To simplify the notation, we neglect the time argument of the function $N(t)$ and write N. The growth rate $r(t)$ is an unknown function in the variable t, which we will estimate using linear and non-linear regression.

The data for varying growth rate for world population are shown in Table 1 and displayed in Figure 1. The source is [1].

r_0	0.0207	0.0197	0.0179	0.0180	0.0181	0.0153
World pop	3,700,437,046	4,079,480,606	4,458,003,514	4,870,921,740	5,327,231,061	5,744,212,979
Year	1970	1975	1980	1985	1990	1995

r_0	0.0133	0.0126	0.0124	0.0119	0.0105
World pop	6,143,493,823	6,541,907,027	6,956,823,603	7,379,797,139	7,794,798,739
Year	2000	2005	2010	2015	2020

Table 1. Population and varying growth rate for world population over time. Source [1].

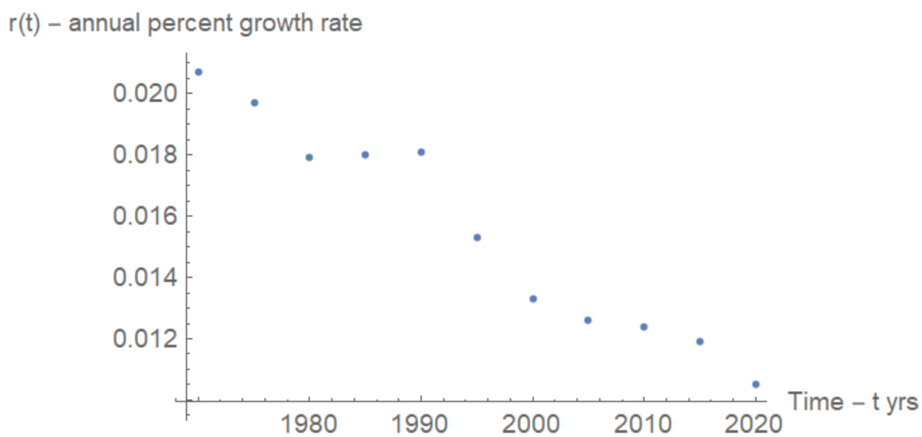

Figure 1. Plot of the changing annual world population growth rates, $r(t)$ vs. t, time in years.

Question 2. Assume the growth rate for world population is constant, say r. From the data in Table 1 offer an estimate for r. Create a model for the world population for the future. What does the future hold for the world?

If we believe, based on the data, that the rate at which the world's population grows is NOT constant then we attempt to create a function $r(t)$ for the changing growth rate, certainly based in what we see in Figure 1. We consider the following variations.

Question 3. Offer a rationale for each of the two candidates for $r(t)$ the changing growth rate for world population.

(a) $r(t) = a_0 + a_1 t$,

(b) $r(t) = a_0 + a_1 t + a_2 \sin\left(\frac{2\pi t}{L} + a_3\right)$.

where $L \approx 30$ indicates the average age of the mother at the birth of one of the offspring.

Question 4. Show the solution to (3) is (4).

$$N(t) = N(t_0) \cdot e^{\left(\int_{t_0}^{t} r(\tau) d\tau\right)}. \tag{4}$$

We will assume the present year, $t_0 = 2019$, when there are about 7.714 billion inhabitants on

Earth, i.e. $N(2019) = 7.714$ billion.

First model for $r(t)$

First we consider the model, $r(t) = a_0 + a_1 t$.

Question 5. Show that if the growth rate data is fitted to this simple regression line, the population model takes the form as seen in (5).

$$N(t) = N(t_0)\frac{e^{a_0 t + a_1 \frac{t^2}{2}}}{e^{a_0 t_0 + a_1 \frac{t_0^2}{2}}}. \tag{5}$$

Question 6. To estimate the parameters a_0 and a_1 use a least squares method to obtain values for the two parameters, a_0 and a_1. Use these obtained parameter values for a_0 and a_1 in (5) and then plot your model for the world's population through 2200.

The resulting world population model should look something like what appears in Figure 2. Does yours look like this? Explain what this plot is showing. Is this reasonable? Why or why not? Based on the plot of the world's population in Figure 2 determine when the population will peak or reach a maximum.

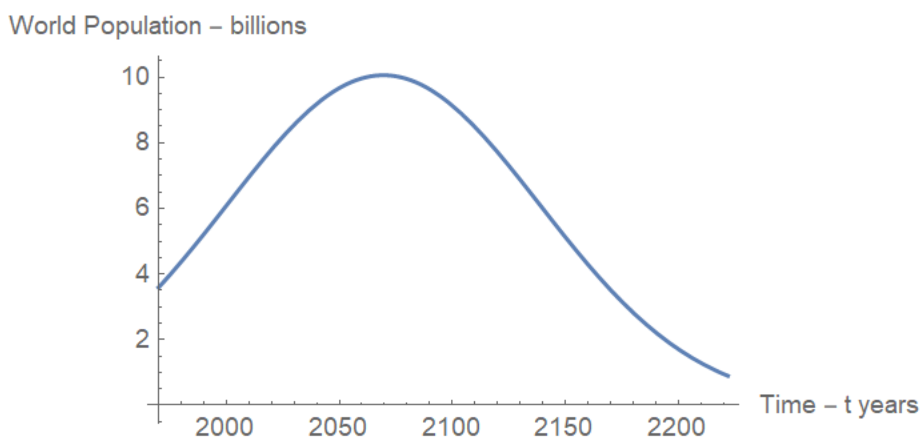

Figure 2. Plot of the world's population using growth rate (a) $r(t) = a_0 + a_1 t$ with best values of a_0 and a_1.

Second model for $r(t)$

Next we consider the model $r(t) = a_0 + a_1 t + a_2 \sin\left(\frac{2\pi t}{L} + a_3\right)$

Question 7. Show that if the growth rate data is fitted to this more complicated function for $r(t)$, the population model takes the form as seen in (6).

$$N(t) = N(t_0)e^{\left(-\frac{-4\pi a_0 t + 8076\pi a_0 - 2\pi a_1 t^2 + 8152722\pi a_1 + 60 a_2 \cos\left(a_3 + \frac{\pi t}{15}\right) + 15\sqrt{2\left(5+\sqrt{5}\right)}a_2 \sin(a_3) + 15\left(\sqrt{5}-1\right)a_2 \cos(a3)}{4\pi}\right)}. \tag{6}$$

Question 8. To estimate the parameters a_0, a_1, a_2, and a_3 use a least squares method to obtain values for the four parameters, a_0, a_1, a_2, and a_3. Use these obtained parameter values for a_0, a_1, a_2, and a_3 in (6) and then plot your model for the world's population through 2200.

The resulting world population model should look something like what appearss in Figure 3. Does yours look like this? Explain what this plot is showing. Is this reasonable? Why or why not? Based on the plot of the world's population in Figure 3 determine when the population will peak or reach a maximum.

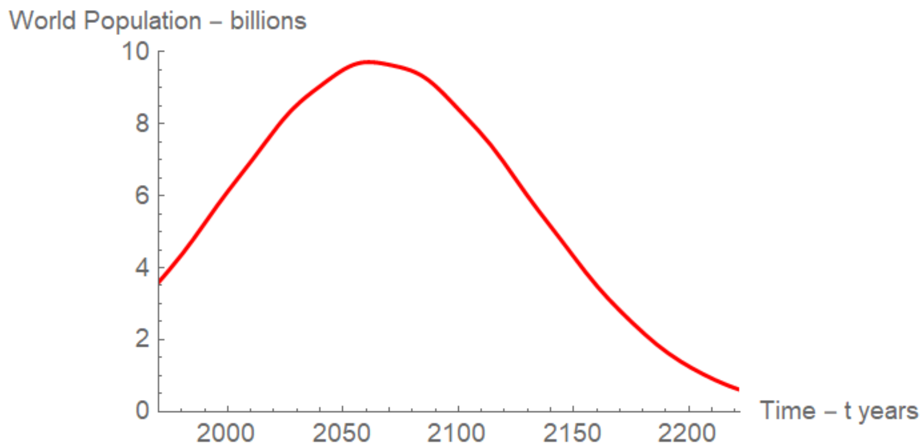

Figure 3. Plot of the world's population using growth rate (b) $r(t) = a_0 + a_1 t + a_2 \sin\left(\frac{2\pi t}{L} + a_3\right)$ with your best values of a_0 and a_1.

Question 9. You should conjecture and support several other functions for $r(t)$ and apply a similar analysis as we do here to predict the future world populations.

Question 10. Finally, compare the future for world population as depicted by all four models for $r(t)$: constant growth rate, Model (a), Model (b), and your additional model. What do you think about the future of the world's population based on these analyses?

More Models for Consideration

We consider two more models for $r(t)$:

(c) $r(t) = a_0 + a_1 \cdot N(t)$, the logistic model,

(d) $r(t) = a\left(\frac{N(t)}{A} - 1\right)\left(1 - \frac{N(t)}{K}\right)$, Allee effect model.

Model (c), the logistic model, is a model in which the growth coefficient, $r(t) = r(N(t)) = a_0 + a_1 \cdot N(t)$, is expressed as a function of the population size, $N(t)$. This gives the world population growth model of the form seen in (7):

$$\frac{dN}{dt} = r(t) \cdot N(t) = (a_0 + a_1 \cdot N(t)) \cdot N(t)\,. \tag{7}$$

Question 11. We will have to estimate the function of the growth coefficient $r(t) = a_0 + a_1 \cdot N(t)$ by linear regression using the data in Table 1. Do that and show that your best fit coefficients are $a_0 = 0.0293993$ and $a_1 = -0.00245444$.

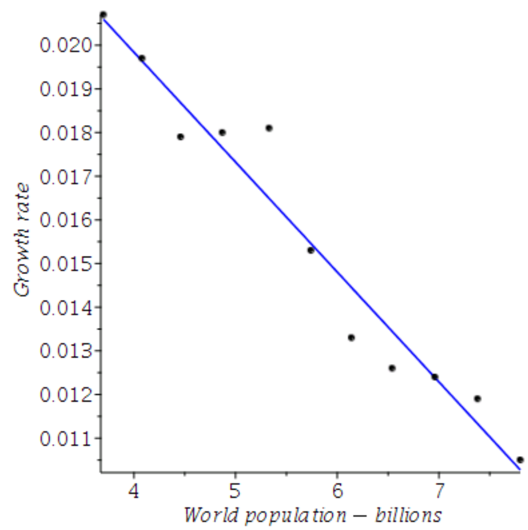

Figure 4. Plot of the growth rate estimation (c) $r(t) = a_0 + a_1 N(t)$ with the best estimations of parameters a_0 a a_1.

Question 12. After substituting Model (c) $r(t) = a_0 + a_1 \cdot N(t)$ into the model differential equation (3) and solving this differential equation for the function $N(t)$ verify that we obtain:

$$N(t) = \frac{9.75874 \times 10^{23}}{e^{0.0293993t} + 3.31938 \times 10^{25}}\,. \tag{8}$$

Question 13. Demonstrate the fact that the graph of the world's human population should "level off" to 11.8 billion.

The model will never be declining because we chose the declining function $r(N(t))$ (straight line) and thus, the larger the population, the slower it will grow. The smaller the growth coefficient, the smaller the population growth - so growth will be slower, but it will never stop altogether. Such a model can be used, for example, to approximate the capacity of the environment, in our case around 11.8 billion people.

As for Model (d), "Although the concept of Allee effect had no title at the time, it was first described in the 1930's by its namesake, Warder Clyde Allee. Through experimental studies, Allee was able to demonstrate that goldfish grow more rapidly when there are more individuals within

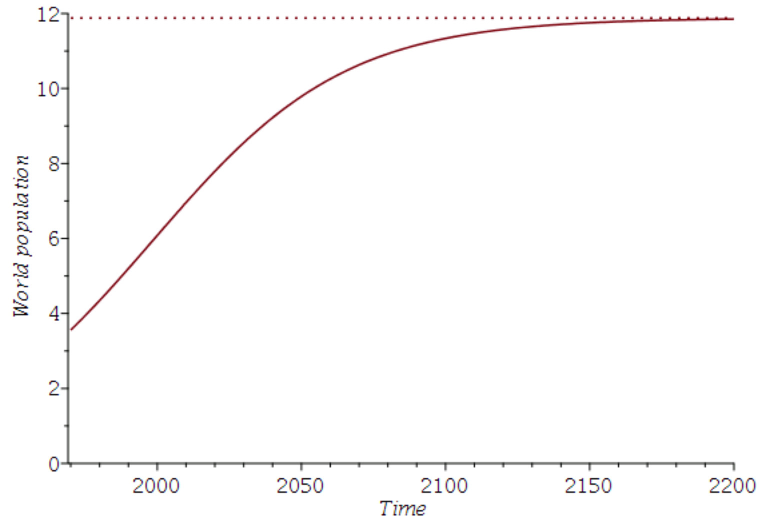

Figure 5. Plot of the world's population using growth rate (c) $r(t) = a_0 + a_1 N(t)$ with the best values of a_0 and a_1.

the tank."[4] This led him to conclude that aggregation can improve the survival rate of individuals, and that cooperation may be crucial in the overall evolution of social structure. The Allee effect model, has a as the intrinsic rate of increase, K as the carrying capacitiy of the environment, and A as a critical point or Allee threshold.

We read, "This graph is an example of a strong Allee effect where the population has a negative growth rate for $0 < N(t) < A$ and are eventually driven to extinction, and a positive growth rate for $A < N(t) < K$ (assuming $0 < A < K$). In cases of a weak Allee effect, populations below the threshold are merely hampered in their rates of growth. The occurrences of Allee effect are more easily observed from small populations as populations above the threshold follow a logistic growth. Since human population has never dropped to such a low level, there is no research concerning about influence of Allee effect to the human population, but Allee effect has been observed in many species, especially animals that hunt for prey or defend against predators as a group."[3]

From a mathematical point of view, we need to estimate $r(N)$ as a quadratic function of N which can be done by using data beyond this threshold[1]. However, Table 1 does not contain enough information for such an estimation, hence it is necessary to use all 70 measurements of world population size from 1951 to 2020. These data can be found in [2]. Quadratic growth rate $r(N)$ leads to a world population growth model of the form seen in (9).

$$\frac{dN}{dt} = a\left(\frac{N(t)}{A} - 1\right)\left(1 - \frac{N(t)}{K}\right) \cdot N(t)\,. \tag{9}$$

Question 14. The function of the growth coefficient $r(t) = a_0 + a_1 \cdot N(t) + a_2 \cdot (N(t))^2$ can be

[1] The threshold is the value, at which a parabola has its maximum.

estimated by linear regression using the data in [2] or the data in file population.xlsx. Try it and show that the best fit coefficients are $a_0 = 0.016645$, $a_1 = 1.938243 \cdot 10^{-12}$ and $a_2 = -3.693209 \cdot 10^{-22}$. Compare your results with those depicted in Figure 6.

Question 15. After substituting Model (d) $r(t) = a_0 + a_1 \cdot N(t) + a_2 \cdot (N(t))^2$ into the model differential equation (3) use a numerical method to estimate a solution $N(t)$ of the obtained equation with initial condition $N(2019) = 7.714$. Verify that your result agrees with Figure 7.

Question 16. Demonstrate the fact that the graph of the world's human population should "level off" to 9.83 billion when using Model (d).

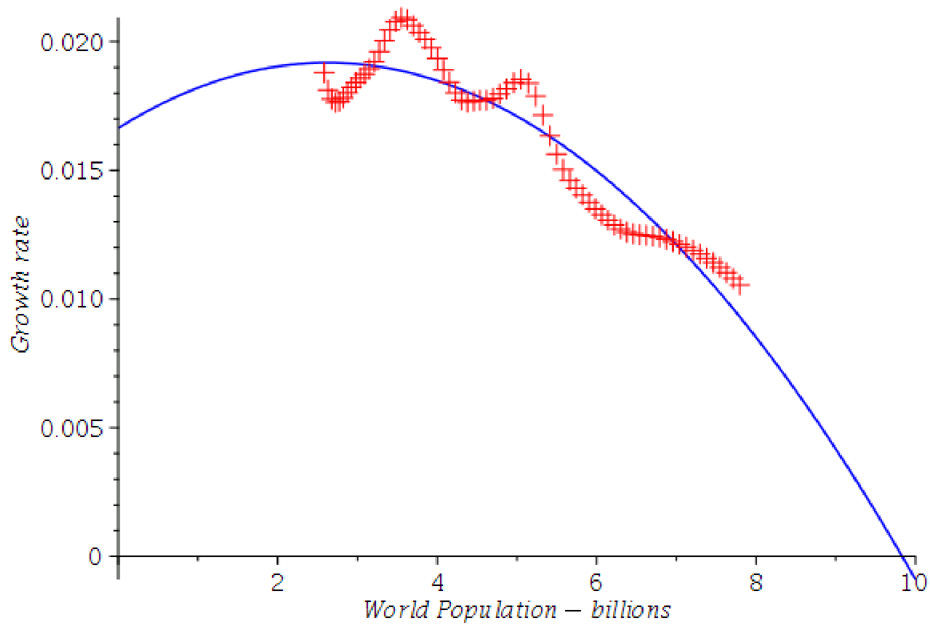

Figure 6. Plot of the growth rate estimation (d) $r(t) = a_0 + a_1 \cdot N(t) + a_2 \cdot (N(t))^2$ with the best estimations of parameters a_0, a_1 and a_2.

REFERENCES

[1] Current World Population. https://www.worldometers.info/world-population. Accessed 24 January 2020.

[2] World Population by Year. . Accessed 24 January 2020.

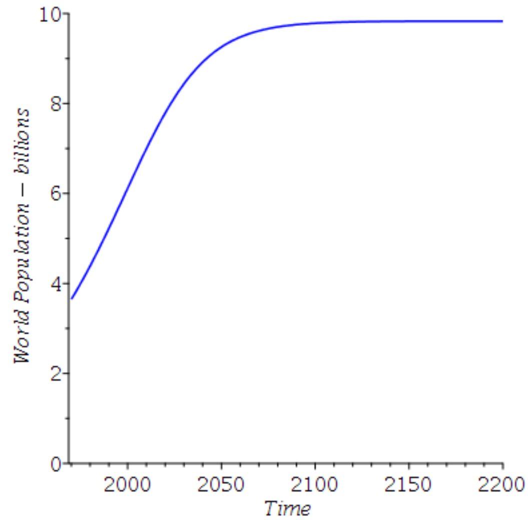

Figure 7. Plot of the world's population using growth rate (d) $r(t) = a_0 + a_1 \cdot N(t) + a_2 \cdot (N(t))^2$ with the best values of a_0, a_1 and a_2.

[3] Project Rhea. 2020. Logistic Models. https://www.projectrhea.org/rhea/index.php/Logistic_Models. Accessed 28 January 2020.

[4] Wikipedia contributors. (2020, January 25). Allee effect. In *Wikipedia, The Free Encyclopedia*. https://en.wikipedia.org/w/index.php?title=Allee_effect&oldid=937563935. Accessed 28 January 2020.