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STUDENT VERSION
WATER FLOWING FROM TANK

Brian Winkel
Director SIMIODE
Cornwall NY USA

Abstract: We help students develop a model (Torricelli's Law) for the height of a falling column of water with a small hole in the container at the bottom of the column of water through which water exits the column. We offer several sources of simulations on YouTube from which one can collect data and ask students to verify their model through parameter estimation.

Keywords: flow, column tank, aperture, Torricelli's Law, data

Tags: water, first order, differential equation, Conservation of Energy, parameter estimation

STATEMENT

In Figure 1 we see a screen capture from a video of an experiment [1] to display and measure the height (cm) of a column of water in a graduated cylinder at time (s) when there is a tiny hole at the bottom of the cylinder through which the water exits. The time elapsed is 29 s while the water level is at 17.5 cm. Table 1 shows sample data from the same run of the experiment in which the column of water was 35 cm high originally, i.e. at $t = 0$.

Derivation of Model from First Principles

We examine a model based on first principles, not just on empirical fit. Given a column of water in a cylinder with a tiny hole at the bottom of the cylinder through which the water exits, how long does it take to empty the cylinder down to the level of the hole? More specifically, if we are given the cross sectional area of the cylinder as a function of height and the area of the tiny hole at the bottom of the cylinder can we model the outflow of the water from the cylinder?

Finally, we offer a plot (Figure 2) of the data in Table 1.

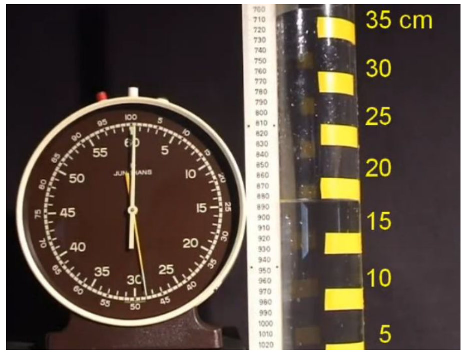

Figure 1. Screen capture [1] from a video of an experiment to display and measure the height (cm) of a column of water in a graduated cylinder at time (s) when there is a tiny hole at the bottom of the cylinder through which the water exits.

Time (s)	Height (cm)
0	35
8	30
16	25
25	20
35.5	15
48.7	10
66.8	5

Table 1. Time vs. height data for column of water experiment (FourthTrial) from [1].

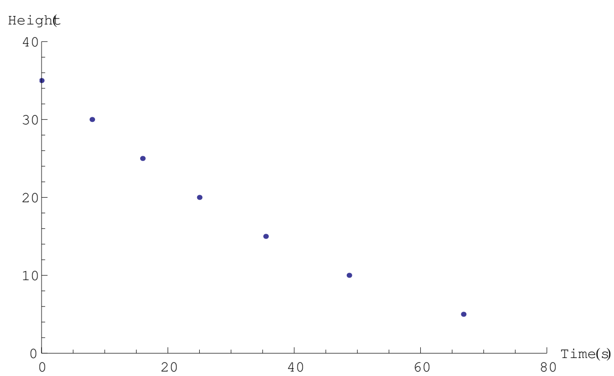

Figure 2. Plot of the data in Table 1.

(1) Form a mathematical model for the height of water in the column in Table 1 and compare it to your data. How good is your model? Your physics? What measure will you use?

Now for some basic physics background. We first consider a law of physics which can help us. The Law of Conservation of Energy says that the sum of the potential energy and the kinetic energy of a particle of mass m is constant, i.e. total energy is conserved. So if we consider a particle of water of mass m initially atop a cylinder of water, some h meters above a small, sharp-edged opening in the side wall of the cylinder through which the water can exit the cylinder, this mass of water has initial potential energy $PE_i = m \cdot g \cdot h$, where g is the acceleration due to gravity, and initial kinetic energy $\frac{1}{2}mv_i^2$ (KE_i), where v_i is the initial velocity of the mass. Thus, we have an initial total energy of TE_i when the mass of water is on the top of the cylinder of water:

$$TE_i = KE_i + PE_i = \frac{1}{2}mv_i^2 + mgh\,.$$

When this mass of water reaches the opening it has height 0 meters above the opening and a final velocity of v_f. Hence, the total energy at the final time (TE_f) the mass reaches the opening is

$$TE_f = KE_f + PE_f = \frac{1}{2}mv_f^2 + mg \cdot 0 = \frac{1}{2}mv_f^2\,.$$

Now, by the Law Conservation of Energy we know $TE_i = TE_f$ and thus we have

$$\frac{1}{2}mv_i^2 + mgh = \frac{1}{2}mv_f^2\,,$$

which when one solves for v_f yields $v_f = \sqrt{2gh + v_i^2}$. In particular, if our element of mass of water was at rest initially, i.e. $v_i = 0$ we have the classical Torricelli's Law

$$v_f = \sqrt{2gh}\,,$$

where v_f is the speed of the element of mass of water as it exits the opening in the cylinder.

In particular, we can use this velocity of an element of water at the bottom of the cylinder given the height of the colulmn of water in the cylinder, h, to do some accounting in the change of the volume of the column of water in the tank at time t and hence the height of the column of water, $h(t)$, in the tank at time t.

We equate two ways of computing the change in the volume of water in the tank at time t.

$$A(t)h'(t) \tag{1}$$

where $A(h(t))$ is the cross sectional area of the column at height $h(t)$, often constant, so $A = A(h(t))$ in our case, and

$$v_f \cdot a \cdot \alpha = \sqrt{2gh} \cdot a \cdot \alpha \tag{2}$$

where v_f is the velocity of the water exiting the column of water when the water is at height $h(t)$, a is the cross section of the small bore hole through which the water exits, and α is an empirical number which indicates the perentage of the maximum flow rate which actually gets through the small bore hole, the reduction due to friction and constriction in the small hole. α, is called the *discharge or contraction coefficient*.

Now equating (1) and (2) gives rise to a working version of Torricelli's Law for the height of a constant cross section column of water, $h(t)$, at time t:

$$A(h(t)) \cdot h'(t) = -a\alpha\sqrt{2gh(t)}\,, \tag{3}$$

where $h(t)$ is the height of the column of water at time t. Note the negative sign on the right hand side of (3) as water is leaving the column of water.

If we solve for $h(t)$ and note that $A(h(t)) = A$, while gathering all the constants except g into one big constant, b, we have

$$h'(t) = -b\sqrt{gh(t)} \tag{4}$$

(2) Solve (4) for $h(t)$ if we know the initial height of the column of water, i.e. $h(0) = h_0$ and we presume the cross sectional area of the column of water at height h is constant.

Modeling the outflow from a cylinder of water

Suppose we have a cylinder of height 0.20 m and radius 0.05 m and it is filled with water. There is an opening in the side of the cylinder at the bottom of the cylinder in the form of a sharp edged circle of radius $r = 0.01$ m.

Empirically, it is known that while the area of the opening is $\pi r^2 = \pi(.01)^2$ m^2 the effective area for fluid flow through the opening is only a fraction of the area. That fraction, α, is called the *discharge or contraction coefficient*. Let us assume $\alpha = 0.7$, which is a reasonable value from the literature for various opening shapes and sizes.

We now build a differential equation to determine the height of the water in the cylinder as a function of time, $h(t)$ in meters – in this case. To do this we need to use accounting!! This will prove to be a nice way to build models, namely compute some quantity in two different ways and set the two different computations equal to each other. If the computations involve derivatives we have built a differential equation!

So let us compute, in two different ways, the rate at which the volume of water changes at time t. We shall do this for a general cross sectional area of our cylinder as a function of height, $A(h(t))$.

(L) At time t a small element of height changed, $h'(t)$, over a cross sectional area $A(h(t))$. Hence we will have lost

$$A(h(t)) \cdot h'(t) \tag{5}$$

volume of water. In our case, for our circular cylinder of constant radius 0.05 m, we have the following computation:

$$A(h(t)) \cdot h'(t) = \pi(.05)^2 h'(t) \tag{6}$$

(R) Since the water is flowing out of the opening with a velocity (according to Torricelli's Law) of $\sqrt{2gh(t)}$ m/s and our opening's cross sectional area is $a = \pi r^2 = \pi(.01)^2$ m^2 then we are losing water at a rate of

$$\alpha a\sqrt{2gh(t)} = \alpha\pi r^2\sqrt{2gh(t)} = .7\pi(.01)^2\sqrt{2gh(t)} \quad \text{m}^3/\text{s}\,, \tag{7}$$

where we recall the effective area for fluid flow through the opening is only a fraction ($\alpha = .7$) of the area.

Now if we equate the computations for the loss of water found in (5) with that computed in (7) we obtain the following differential equation (adding our initial height $h(0) = h_0 = 0.20$ m as an initial condition):

$$A(h(t)) \cdot h'(t) = \alpha\, a\sqrt{2gh(t)}\,, \quad h(0) = h_0 \tag{8}$$

in general, and in our case, using $g = 9.8$ m/s^2,

$$\pi(.05)^2 h'(t) = .7\pi(.01)^2\sqrt{2(9.8)h(t)}\,, \quad h(0) = 0.20\,. \tag{9}$$

We note that we could collect all the constants in (9) into one constant b outside the square root sign and write:

$$h'(t) = b\sqrt{g \cdot h(t)}\,. \tag{10}$$

Here $b = \dfrac{.7\pi(.01)^2}{\pi(.05)^2}\sqrt{2} = 0.039598$.

(3) Solve the differential equation (10) (be sure you pick the correct branch of the solution) and determine how long it takes to empty the cylinder of water. Plot the height $h(t)$ as a function of time, t, for the duration of the draining.

(4) Suppose $\alpha = 0.6$ instead of 0.7, then how long would it take to empty the cylinder of water? Plot the height $h(t)$ as a function of time, t, for the duration of the draining. Compare this answer and the plot with those from (a). Explain why this is reasonable.

(5) Continuing with (3), determine the velocity of the surface of the water and plot it over the duration of the draining. Explain why what you see is reasonable.

Now let us play some "what if" games with our model.

(6) Vary the radius of the *opening*, r, and determine the time $T(r)$ at which $h(T(r)) = 0$ for each r, i.e. the time $T(r)$ to empty the tank as a function of r. Plot $T(r)$ and comment on its reasonableness. Use $\alpha = 0.7$ and the same parameters as in the discussion above.

(7) Vary the radius of the *cylinder*, r, in (3) and determine the time $TT(r)$ at which $h(TT(r)) = 0$ for each r, i.e. time $TT(r)$ to empty the tank as a function of r. Plot $TT(r)$ and comment on its reasonableness. Use $\alpha = 0.7$ and the same parameters as in (3) above.

(8) Suppose the tank of water were in the shape of a frustum of a right cone, i.e. we change the radius of the bottom circular base, call it s, out (or in!) from the original radius. Again we have a cylinder of water of height $h_0 = 0.2$ m, a radius of 0.05 m at the top of the cylinder, an opening of radius 0.01 m at the bottom of the cylinder, but the base of the cylinder is now a circle with radius s m. Use $\alpha = 0.7$ here, also. Collect some data on how long it takes to empty the cylinder for s values in the region $s \in [0, 0.2]$, say, and try to come up with a function that models the time it takes to drain the cylinder as a function of the bottom radius, s. NB: If $s = 0.2$ then we have already done this analysis, as this is the constant cross sectional area case. Explain why your obtained function offers a reasonable relationship. Incidentally, if $s = 0$ we have an inverted cone.

Two Sources of Data For Model Confirmation

One could collect data with a modest apparatus of a graduated cylinder with a tiny hole near the bottom and a clock, all in the video frame at once to see height and time at the same time. However, we go to several sources on the web for videos and we collect data.

One source for data is from the University of Alicante physics experiments [1]. At the 3:41 mark of the 10:58 video there is a fourth run of data collection of water running out of a graduated cylinder of circular constant cross section with radius 1.68 cm. The initial height of the liquid in the tube is $h = 35$ cm.

(9) Here is the data we collected from our first video source [1]. How does it compare to yours? How well can you model this data set? Offer up analysis to determine parameters from your model. Compare data with model.

Time (s)	Height (cm)
0	35
8	30
16	25
25	20
35.5	15
48.7	10
66.8	5

Table 2. Time vs. height data for column of water experiment (FourthTrial) from [1].

A second source for data is the video [4]. Here, we see four trials of data collection of water running out of a graduated cylinder of constant circular cross section. The data comes from the FourthTrial which begins at 5:34 of the 7:23 video [4]. We can also obtain the radius at the 9:34 time in to a second video [5]. There is a picture which enables us to estimate the diameter of the cylinder as 19 cm or a radius of 9.5 cm.

(10) In Table 3 there is the data we collected from our second source [4]. How does it compare to yours? What data did you get from the First, Second, and Third Trial? How well can you model each data set? Offer up analysis to determine parameters from one of the models above. Compare data with model.

Additional Questions to Consider

Here are some additional questions from [3].

(11) Suppose we have a cylindrical tank with the area of the tiny hole at the bottom being a and cross sectional area of the cylinder being A.

Time (s)	Height (cm)
0	13
.58	12
1.35	11
1.95	10
2.85	9
3.65	8
4.55	7
5.55	6
6.55	5
7.55	4
8.55	3
10.45	2
13.45	1

Table 3. Time vs. height data for column of water experiment (FourthTrial) from [4].

a) How long does it take to empty the tank?

For each of the following physical changes in the tank, describe the effect on the time it takes to empty the tank.

b) Double the area, a, of the tiny hole at the bottom of the cylinder.

c) Change g by considering the tank to be on the Moon. There gravity is one-sixth that of the Earth.

Building a Water Clock

(12) Design a cylinder of cross-sectional area $A(h)$ at height h so that the water falls at a constant rate? In this way, we could display equal intervals, say quarter hour, in equal vertical distance, marking the time as it passes. To model a constant rate of fall for the height, $h(t)$, of water we need to set $h'(t) = c$ for some constant we desire, e.g., 1 centimeter per 15 minutes. This clock cylinder could then be marked at vertical heights of 1 cm with the knowledge that for ever centimeter the water falls 15 minutes has elapsed.

So, what shape must we elect; namely what is the cross sectional area at height h, i.e. $A(h)$, which actually gives us a constantly decreasing height in our water level?

When done you can run right down to your 3D printer shop and make your own water clock.

Video sources for data collection

We summarize our data sources.

We used two videos of columns of water falling, without knowledge given as to the radius of the opening at the bottom of the cylinder. These were from YouTube and we cite them properly below. These are referred to in Activity 13.

We also built experimental apparatus to collect data using a free digital clock on a computer (`www.on-line-stopwatch.com`) which measures and display nicely and in large fonts real time to the thousandth of a second, a clear sparkling water plastic container, a collection bin for the exiting water, and an iPad to produce video of the experiments. We did this for small holes at the bottom of the cylinder of diameters 13/16", 9/64", 11/64", and 7/64". In the latter case we used two data sets - sampling from the entire run and then sampling from a smaller section of the run. We placed the following videos for your use in this section for this scenario 1-15-Torricelli on YouTube. These are referred to in Activity 14.

Source 1:

Name: Ley de Torricelli Vaciado de un depósito
http://www.youtube.com/watch?v=6LOgdOfFAbI

Source 2:

Name: Numec project Sem1 2011 THE Massooraahs!
http://www.youtube.com/watch?v=44fZdi7QC8E

Sources 3 - 6 are available at YouTube and at this Modeling Scenario site.
https://www.simiode.org/resources/488

Source 3:

Name: 13Over64Inch Small Hole
https://www.youtube.com/watch?v=gsNdsuQ1ZCo
https://www.youtube.com/watch?v=SCNqxFe3eIs
Also available at https://www.simiode.org/resources/488/video?resid=2085.

Source 4:

Name: 9Over64 Inch Small Hole
https://www.youtube.com/watch?v=gsNdsuQ1ZCo
and https://www.youtube.com/watch?v=6cUuNUjlaPI
Also available at https://www.simiode.org/resources/488/video?resid=2083.

Source 5:

Name: 11Over64Inch Small Hole
https://www.youtube.com/watch?v=76diQ1eW8i4
and https://www.youtube.com/watch?v=NBr4DOj4OTE
Also available at https://www.simiode.org/resources/488/video?resid=2087.

Source 6:

Name: 7Over64 Inch Small Hole

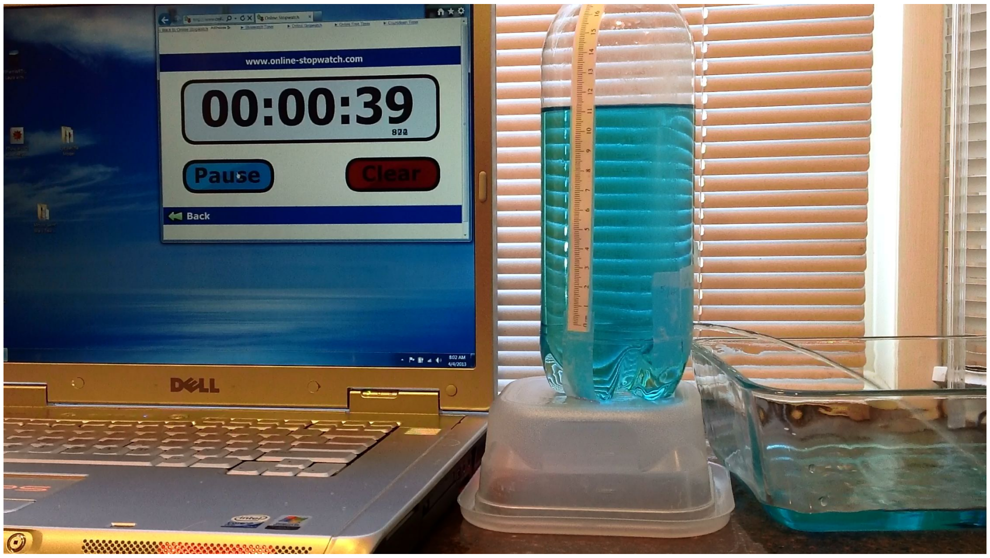

Figure 3. Experimental apparatus for collecting data on water flowing from cylindrical column of water. Left to right we see laptop computer with `www.online-stopwatch.com` running to get time to the nearest thousandth of a second, marked circular cylinder of water (in cm of height) in the form of a sparkling water plastic container, water exiting to the right from small hole of radius 9/64 ", and receiving container for water leaving the cylinder.

https://www.youtube.com/watch?v=k7TSAqRfHgg
and https://www.youtube.com/watch?v=xDyDSPydN_E.
Also available at https://www.simiode.org/resources/488/video?resid=2084.

(13) and (14) Using one of the videos from Source 1 or Source 2 and then one of the videos from Sources 3-6 collect data on time and height of the column of water and model it, first with a Power Rule and then if possible with Torricelli's Law differential equation. Estimate all parameters possible, and confirm your model against the data you have collected.

Beginning and End Observations - You Mess with Mr. In Between

(15) A small hole of radius $s = 0.138906$ cm is drilled in the side of a right circular cylindrical container of radius $r = 4.16986$ cm and the height of the water level (above the hole) goes from 12.5 cm at a stop watch time of 1:40.419 min to 3 cm at a stop watch time of 2:34.116 min. Use two different models to estimate the height at intermediate times. These models are (1) linear model: $h'(t) = -kh(t)$, $h(0) = 12.5$ with $h(143.286) = 2.0$ and (2) Torricelli's Law: $Ah'(t) = -\alpha \cdot a \cdot \sqrt{2 \cdot g \cdot h(t)}$, $h(0) = 12.5$ with $h(143.286) = 2.0$. Here A is the cross sectional area of the cylinder and a is the area of the small hole for drainage, while α is the discharge or contraction coefficient.

After you have completed your analyses, go to the video, entitled 11Over64 Inch Small Hole from the from this Scenario - 1-15-Torricelli, and compare your observations from the video with those from each of your model. Which model is the better one? Why?

We offer a table (Table 4) of clock times, relative times, and known observations. You fill in the rest from your modeling activity results.

Clock Time t (min)	Relative Time t (s)	Linear h (cm)	Torricelli h (cm)	Observed h (cm)
1:40.419 = 100.419	0	12.5	12.5	12.5
1:45.598 = 105.598	5.179			
1:47.568 = 107.788	7.369			
1:49.623 = 109.523	9.104			
1:50.985 = 110.985	10.566			
1:56.388 = 116,388	15.969			
2:02.057 = 122.057	21.638			
2:08.032 = 128.032	27.613			
2:16.386 = 136.386	35.967			
2:25.568 = 145.568	45.149			
2:34.116 = 154.286	53.867			2.0

Table 4. Clock Time and Relative Time for sample Torricelli Law run.

REFERENCES

[1] University of Alicante. 2013. *Experienceas de Física. Demostraciones y Prácticas de Laboratorio*. http://www.dfists.ua.es/experiencias_de_fisica/index1.html. Also available as *Ley de Torricelli Vaciado de un depósito* at http://www.youtube.com/watch?v=6LOgdOfFAbI. Accessed 30 March 2013.

[2] Farmer, T. and F. Gass. 1992. Physical Demonstrations in the Calculus Classroom. *The College Mathematics Journal*. 23(2): 146-148.

[3] Krieder, D., D. Lahr, and S. Diesel. 2005. *Principles of Calculus Modeling: An Interactive approach*. From the Quiz in Unit on Torricelli's Law. http://www.math.dartmouth.edu/ klbooksite/2.18/218quiz/218quiz_p02.htm.

[4] Masooraahs. 2011. Numec project Sem1 2L011 THE Masooraahs! http://www.youtube.com/watch?v=44fZdi7QC8E. Accessed 31 March 2013. Video length 7:23.

[5] Masooraahs. 2011a. Numerical methods extended video. http://www.youtube.com/watch?v=-HsVq3KJIGA. Accessed 31 March 2013. Video length 9:58.