STUDENT VERSION
WATER FLOWING OUT FROM COLUMN OF WATER
SALT WATER FLOWING IN

STATEMENT

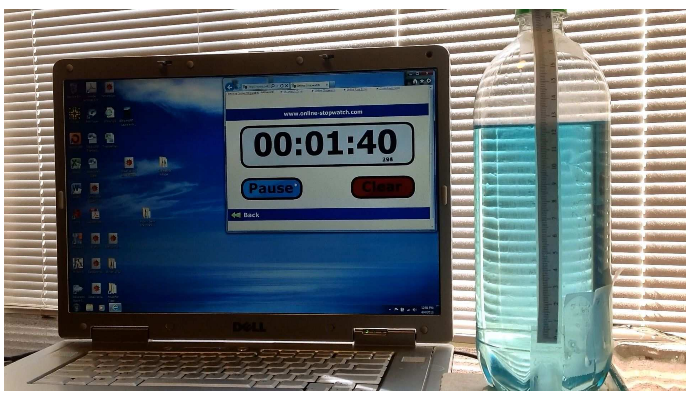

Figure 1. Screen capture from a video of an experiment to display and measure the height (cm) of a column of water in a graduated cylinder at time (s) when there is a tiny hole at the bottom of the cylinder through which the water exits.

In Figure 1 we see a screen capture from a video of an experiment found on YouTube [3] to display and measure the height (cm) of a column of water in a graduated cylinder at time (s) when there is a small hole at the bottom of the cylinder through which the water exits. Table 1 shows sample data from the run of the experiment in which the column of water was 6.65 cm high originally, i.e. at $t = 0$. The cross sectional radius of the circular cylinder is $r = 4.16986$ cm (measured by dividing the circumference by 2π) and the radius of the small hole at the bottom of the cylinder is 13/64 inches or $s = 0.257969$ cm.

We collect data from the video. Basically, the YouTube viewer stops the video and records the time to the nearest thousandth of a second as offered by the on-screen timer and the height to the nearest 0.05 cm. The latter is done visually from the stopped video screen shot.

Time (s)	Height (cm)
0.000	6.65
0.996	6.35
1.760	6.10
2.197	6.00
3.164	5.70
3.461	5.60
4.040	5.45
4.740	5.25
5.364	5.05
6.596	4.70
8.094	4.35
9.794	3.95
11.940	3.45
13.367	3.15

Table 1. Time vs. height data collected by the author for column of water experiment. The times are given to a thousandth of a second because when the video is stopped to sample data the digital clock renders the time in thousandths of a second. Whereas the height at each point is estimated from the screen by the viewer to the nearest 0.05 cm.

We offer a plot of the data in Table 1.

Using Torricelli's Law [2] and [1, pp. 2-4] a mathematical model [1, Equation (5) on p. 4] for the height of water, $h(t)$, in the column offered in the data in Table 1 can be built:

$$A(h(t)) * h'(t) = -\alpha\, a \sqrt{2gh(t)}\,, \quad h(0) = h_0\,. \tag{1}$$

Here $A(h(t))$ is the cross sectional area in cm^2 of the column of water at height $h(t)$. In our case $A(h(t)) = A = \pi(4.16986)^2$ at all heights as our cylinder is constant cross sectional circular area. a is the area of the small hole. In our case $a = \pi r^2 = \pi(0.257969)^2$ cm^2. g is the acceleration due to gravity in cm/s^2. At our site for collecting the data $g = 980.3$ cm/s^2.

Empirically, it is known that while the area of the opening of the small hole is $a = \pi r^2 = \pi(0.257969)^2$ cm^2 the effective area for fluid flow through the opening is only a fraction of the area. That fraction, α, is called the *discharge or contraction coefficient*. When we fit the above Torricelli's Law model (1) to the data in Table 1 we obtain a value of $\alpha = 0.716054$, which is a reasonable value from the literature for various opening shapes and sizes. See [1] for guidance and ideas in this regard.

What we have thus far is a differential equation model (2) based on Torricelli's Law which describes the height of the falling column of water in a cylinder of constant circular cross sectional

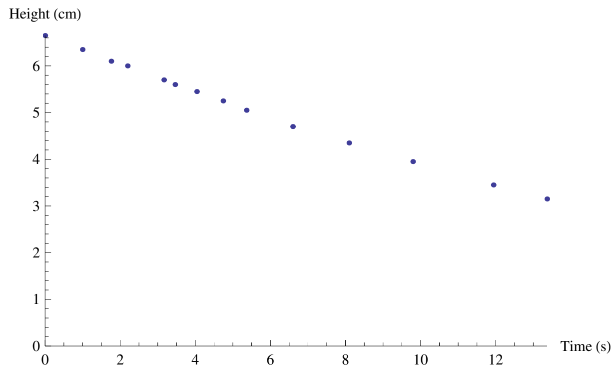

Figure 2. Plot of the data in Table 1.

area:

$$\pi r^2 * h'(t) = A(h(t)) * h'(t) = -\alpha\, a\sqrt{2gh(t)} = -\alpha\, \pi s^2 \sqrt{2gh(t)}\,, \quad h(0) = h_0 \tag{2}$$

with $r = 4.16986$ cm, $s = 0.257969$ cm, and $\alpha = 0.716054$.

Activities

1) Knowing that the volume of the liquid column is $V(t) = \pi r^2 * h(t)$ show that (2) can be written in terms of $V(t)$ instead of $h(t)$ to produce (3). Hint: $V'(t) = \pi r^2 h'(t)$.

$$V'(t) = -\alpha\, \pi s^2 \sqrt{\frac{2gV(t)}{\pi r^2}}\,, \quad V(0) = V_0\,. \tag{3}$$

2) Plot the data from Table 1 and solve the differential equation model (2) for $h(t)$. Then plot your resulting model for $h(t)$ over the data to see how good (REALLY GOOD!) the model is. Determine the time at which all the water has left the column.
3) Convert the data in Table 1 to volume of the column of water as a function of time. Then solve the differential equation model (3) for $V(t)$. Now plot your resulting model for $V(t)$ over the data you just constructed for volume of the column of water as a function of time to see how good (REALLY GOOD!) the model is.

Adding Water to the Column

Suppose now we add water to the top of the column of water at a rate of $Q(t)$ cm^3/s at the same time the water is flowing out the small hole at the bottom.

4) Modify the model in (3) to reflect this addition of $Q(t)$ cm^3/s at the same time as the water is flowing out the small hole at the bottom.
5) Using the parameters from (2) and $Q(t) = 10$ cm^3/s determine the volume of the column of water and plot volume as a function of time in seconds. You should do your plot beyond the observed data, say to time $t = 200$ s. Describe the nature of this plot.

6) For a number of additional values of $Q(t)$, say $Q(t) = 0.1, 2, 4, 8, 15, 20$ cm^3/s determine a relationship between the input flow rate and the eventual equilibrium value of the amount of liquid in the column of water. Could you ascertain this relationship in another, easier, more direct manner? Hint: At equilibrium $V'(t) = 0$.
7) Try some different input flow rates, e.g., $Q(t) = 10 + \sin(0.1t)$ and report out your results on the volume of the column of water. In each case comment on the reasonableness and the expectedness of the results.

Adding Salt Water to the Column

Suppose now we add salt water of concentration $SE(t)$ g/cm^3 to the top of the column of water at a rate of $Q(t)$ cm^3/s at the same time as the water is flowing out the small hole at the bottom. This effectively adds $SE(t) * Q(t)$ grams of salt per second to our column of water while also changing the volume per Activities (4) - (7) above. Further, let us assume the column has a stirrer in place which causes the two different solutions (the water in the column and the water being added to the column) to mix instantaneously. In all cases let us presume that there is no salt in the column of water at the start of any run.

8) Modify the model in (3) to reflect this addition of $Q(t)$ cm^3/s with a concentration of salt of $SE(t)$ g/cm^3 at the same time as the water is flowing out the small hole at the bottom.
9) Using the parameters from (2) and $Q(t) = 10$ cm^3/s with $SE(t) = 0.2$ g/cm^3 determine, $S(t)$, the amount of salt in the column of water and plot this amount as a function of time in seconds. Hint: You will want to construct another differential equation of the form $S'(t) =???$. You should do your plot beyond the observed data, say to time $t = 200$ s. Describe the nature of this plot. Also plot the concentration of salt in the column of water as a function of time over this longer time interval. Is this what you expected? Explain.
10) Consider a number of additional values of $Q(t)$, say $Q(t) = 0.1, 4, 8, 20$ cm^3/s and a set salt concentration in the input flow, say $SE(t) = 0.2$ g/cm^3. In each case determine the long term behavior of the amount of salt in the tank and the concentration of salt in the tank.

REFERENCES

[1] SIMIODE. 2014. 1-15-Torricelli-StudentVersion. `https://www.simiode.org/resources/288`. Accessed 9 September 2014.

[2] Wikepedia. 2014. Torricelli's Law. `http://en.wikipedia.org/wiki/Torricelli's_law`. Accessed 2 January 2014.

[3] Winkel, B. J. 2013. SIMIODE Torricelli 13/64" Small Hole. `http://www.youtube.com/watch?v=e5jLOODNSm4`. Accessed 9 September 2014.