Existence Theorems and their Applications

By

Muhammad Nazam

Department of Mathematics,
Allama Iqbal Open University, H-8 Islamabad, Pakistan.
Email: muhammad.nazam@aiou.edu.pk

https://scholar.google.com/citations?user=aAHRdw0AAAAJ&hl=en&oi=ao

February 13, 2021
Existence Theorems

• A theorem stating the existence of an object, such as the solution to a problem or equation.

• Generally, the existence theorems are of three types:

• (Type 1). Existence theorems that give explicit formulas for solutions, for example:

  Cramer’s Rule:

• Cramer's Rule (studied in linear algebra) is a method that uses determinants to solve systems of equations that have the same number of equations as variables.) It gives a condition for existence of the unique solution to system of linear equations as well as formulas to find this solution.


• Introduction to Line Analysis Algebraic Curves (English)
Existence Theorems

Wronskian: \[ W[f, g] = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} \]

The second order differential equations of the type:

\[ \frac{d^2y}{dx^2} + P(x) \frac{dy}{dx} + Q(x)y = f(x) \quad \ldots (1) \]

where \( P(x), Q(x) \) and \( f(x) \) are functions of \( x \). The variation of parameters method states that:

If \( W[f, g] \neq 0 \), then (1) has exactly two linearly independent solutions. These solutions can be worked out by using Cramer’s Rule.


Existence Theorems

(Type 2). *The theorems which only state the condition(s) for the existence of the objects, for example:*

Bolzano-Weierstrass Theorem:

**Version 1.** Every bounded sequence of real numbers has a convergent subsequence

**Version 2.** Every bounded, infinite set of real numbers has a limit point


Existence Theorems

**Intermediate Value Theorem**

It states that if the real valued function $f$ is continuous on $[a, b]$ and $f(a) < y < f(b)$. Then there exists $a < c < b$ such that

$$y = f(c).$$

[1] O’Connor, John J.; Robertson, Edmund F., "Intermediate value theorem", MacTutor History of Mathematics archive, University of St Andrews
Existence Theorems

Extreme Value Theorem

The extreme value theorem states that if a real-valued function $f$ is continuous on the closed interval $[a, b]$, then $f$ must attain a maximum and a minimum value, each at least once.

Existence Theorems

Mean Value Theorem

It states that:

If the real-valued function \( f: [a, b] \rightarrow \mathbb{R} \) is differentiable over \((a, b)\) and continuous at \( x = a \), and \( x = b \). Then there exists a number \( c \in (a, b) \) such that \( f'(c) \) is equal to the function's average rate of change over the interval, that is

\[
\frac{f(b) - f(a)}{b - a} = f'(c).
\]

Graphically, the theorem guarantees that an arc between two endpoints has a point at which the tangent to the arc is parallel to the secant through its endpoints.
Existence Theorems
Existence Theorems

How do you prove positive derivative means increasing function: with the MVT.

How do you prove derivative 0 on an interval means constant function: with the MVT.

It is the first result which gives an explicit relation between values of $f$ and $f'$.

Curve Sketching, the Fundamental Theorem, Taylor Series, and even Hospital's rules, they all are refined versions of repeated applications of the MVT plus special conditions.

Existence Theorems

Roll’s theorem is a special case of mean value theorem. It states that when \( f(a) = f(b) \) then there exists a number \( c \in (a, b) \) such that \( f'(c) = 0 \).

Peano existence theorem is a fundamental theorem which describe the condition for the existence of the solution to ordinary differential equation with some initial conditions. It states that the differential equations:

\[
y'(x) = f(x, y(x)); \quad y(x_0) = y_0
\]

has a unique solution if \( f \) is continuous.

Existence Theorems

(Type 3) Existence theorems whose proofs involve iteration processes.

Lipschitz continuity

A function $f: \mathbb{R} \to \mathbb{R}$ is called Lipschitz continuous if there exists a positive real constant $k \geq 0$ such that

$$|f(x) - f(y)| \leq k|x - y| \quad \forall x, y \in X.$$

The Picard-Lindelöf theorem, Picard's existence theorem, Cauchy-Lipschitz theorem, or existence and uniqueness theorem gives a set of conditions under which an initial value problem has a unique solution.

Lipschitz continuity is the central condition of the Picard–Lindelöf theorem which guarantees the existence and uniqueness of the solution to an initial value problem.

Existence Theorems

A special type of Lipschitz continuity, called *contraction* is used in the *Banach fixed-point theorem* (an existence theorem). It states that:

*Let* \((X,d)\) *be a complete metric space and* \(T:X \rightarrow X\) *satisfies*

\[
d(Tx,Ty) \leq K \ d(x,y); \ 0 \leq K < 1.
\]

*Then* \(T\) *has unique fixed point in* \(X\).

It is the fundamental existence theorem in the metric fixed-point theory. It appears to be a powerful tool for the existence of the solutions of mathematical models involving differential equations and others representing real world phenomena.

Thank You