Using Mobile Apps to Enhance Learning in Differential Equations

Krista Lucas
Timothy Lucas
Pepperdine University
timothy.lucas@pepperdine.edu

Check out the apps at
http://www.slopesapp.com
http://www.wavespdeapp.com
Class Approach

- Physical assumptions
- Mathematical expressions
- Think before you solve
  - Equilibrium solutions
  - Graphical analysis
  - Slopefields, Phase Planes, etc
- Analytical Solutions
Languages: Swift and Kotlin
Platforms: iOS and Android
http://slopesapp.com (ODE)
iPad Release: Nov 2016
iPhone Release: July 2017
Android Release: Jan 2021
http://wavespdeapp.com (PDE)
iOS Release Date: June 2019
Why Mobile Devices?

- Portable
- Comparatively large screen
- Tactile interface

The Role of iPads in Constructing Collaborative Learning Spaces (Fisher, Lucas and Galstyan, 2013)

Using Slopes to Enhance Learning in Ordinary Differential Equations (K. Lucas and T. Lucas, 2020)
Main Goal: Understand and analyze a mathematical model using techniques learned in class
- Slopefields/Phase Planes, Equilibrium Analysis, Numerical/Algebraic Solutions, ...
- Teams of 3-4 Students
- Final Poster Presentation
  - Judged by math and science faculty
- Images provided by Slopes
Mountain Lions vs. Deer
Three Models Examining Predator Prey Dynamics

Background
Over the last 16 months, Pepperdine has issued 17 warnings regarding mountain lions spotted on campus. In an effort to understand the dynamics behind this rise in sightings (and the Malmö ecosystem in general), we use predator-prey systems of differential equations. Within California, mountain lions feed primarily on deer, and deer are preyed on primarily by mountain lions. Given the omnipresent nature of deer on campus, this likely extends to our local ecosystem. We aim to better understand this relationship, we compare three predator-prey models with increasing complexity.

Basic Model and Coefficient Estimates
In its most nascent form, our model includes two species (mountain lions and deer), and models deer using exponential growth. The variables x and y represent deer (prey) and mountain lion (predator) populations, respectively. The equations are displayed below.

\[ x' = ax - bxy \]
\[ y' = cy - dy + f(x + y) \]

Coefficients:
- a: Rate of growth without predation.
- b: Rate at which predation (interaction) decreases deer population.
- d: Rate at which mountain lions die without prey.
- f: Rate of growth without predation.

While the true values of some parameters are unknown, zoological research can be used to guide many of these choices:
- Deer’s maximum growth rate is estimated using reproduction statistics.
- At any given time, 66% of female deer are pregnant. Deer have an average litter size of 1.9, are pregnant for about 203 days, and exhibit balanced sex ratios.
- If we observed a deer population and returned 203 days later, we would therefore expect to see 62.7% more deer. Therefore, without external constraints the population will grow according to the equation \( x = e^{0.627t} \), with the denoting the starting population, and t denoting change in time (in years).
- The location of equilibria is estimated using fixed intake requirements.
- Should the growth in the deer population produce exactly the amount of meat required to sustain the mountain lion population, no change should occur in either.
- The average mountain lion requires 6.57 pounds of meat per day to survive. As the average mule deer weighs 177 pounds, with at most 133 pounds being edible biomass, a mountain lion’s survival requires 0.049 deer per day, which is 18.0% per year.

First Model Behavior
The phase plane has two equilibrium points: a saddle point at (0,0) and a center at (c/d, a/b). Only in cases of complete extinction and at (c/d, a/b) are both populations at rest. A phase plane is shown below, along with a graph of each population. Coefficients were informed by Coefficients: (as well as reasonable variations thereof), the mountain lions (green) and coyotes (blue) engage in conflict, suppressing the population of each. The mountain lions eventually win, sending the coyotes into extinction. A numeric solver was used to find the equilibrium that all three species approach.

Logistic Growth and Ratio Dependence
Our second model introduces the concepts of logistic growth and density dependence.

The equations are displayed below.

\[ x' = ax - bxy - \frac{h_1xy}{(1 + x + y + gy)} \]
\[ y' = cy - dxy - \frac{h_2xy}{(1 + x + y + gy)} \]

Logistic growth is introduced by the addition of \((1 - kx)\) to the part of the equation controlling deer growth. This takes resource scarcity into account, and ensures that the deer population do not expand beyond its carrying capacity (b). The importance of this feature is highlighted by a potential behavior observable in the first model: without mountain lions the deer population will expand infinitely. While California’s natural carrying capacity for deer is an unexpectedly contentious topic, the deer population peaked at approximately 2 million before declining substantially to its current level of 540,000.

Density dependence is introduced by the denominator now both interaction terms.

The effect of this change is best explained by an example. Say that an environment has 100 deer and 1 mountain lion. Say that a second environment has 10 deer and 1 mountain lions. Under our first model, both interaction terms end up the same: 100 times more interaction coefficient. This is obviously a departure from how deer and mountain lions interact in reality. Thus, the new model makes the effectiveness of predation dependent on this ratio between species. The most effective ratio can be determined using the value the coefficients (in this case just f, as one can always be omitted). For example, if \( f = 1 \), the second environment would yield a larger interaction term under the new model.

To ensure comparability between models, values of coefficients representing biological constants (a, c, r) remain unchanged from the previous model, and interaction coefficients (b, d) are appropriately scaled to adjust for the new term.

Second Model Behavior
The phase plane of the second model is shown below. As the model is not linear, the Jacobian (matrix below) serves as a useful linear approximation.

Equilibrium points: (757, 44), (0,0), (1000, 0)

Determined Coefficients: a = 0.675, b = 0.4, c = 0.82, d = 0.148, k = 1.033, f = 0.12, g = 0.0014, 45896

Equation of the second model:
\[ x' = ax - bxy - \frac{h_1xy}{(1 + x + y + gy)} \]
\[ y' = cy - dxy - \frac{h_2xy}{(1 + x + y + gy)} \]

Adding a Third Species
When compared to empirical data, it becomes clear that both models discussed through this point are incomplete. Just 5,000 wild mountain lions roam California—well below what our two-species model predicts. While a wildlife policy of suppressing dangerous mountain lions while protecting deer from predation is likely the largest culprit, it is also worth considering other interactions within our ecosystem. Our third and final model maintains the concepts discussed in its antecedent while adding a third species: coyotes. The equations are displayed below, with y: denoting mountain lions and y: denoting coyotes.

Third Model Behavior
A plot of all three populations is shown below. Using our starting conditions and parameters (as well as reasonable variations thereof), the mountain lions (green) and coyotes (blue) engage in conflict, suppressing the population of each. The mountain lions eventually win, sending the coyotes into extinction. A numeric solver was used to find the equilibrium that all three species approach.

Equilibrium: The deer population approaches 757, the mountain lion population approaches 1000, and the coyote population approaches extinction.

Conclusion, Sources
While all three models share similar equilibrium points (between mountain lions and deer), behavior around these points differs substantially between models. Furthermore, each addition of complexity made our models more fragile. While both species refused to die in the first model, far more scenarios involved extinction in the third model. This could be due to the nature of natural ecosystems, or the nature of mathematical models.

Logistic Growth and Ratio Dependence

Our second model introduces the concepts of logistic growth and density dependence. The equations are displayed below.

\[ x' = ax \left(1 - \frac{x}{k}\right) - \frac{bxy}{(1 + fx + y)} \]
\[ y' = -cy + \frac{dxy}{(1 + fx + y)} \]

Logistic growth is introduced by the addition of \((1-x/k)\) to the part of the equation controlling deer growth. This takes resource scarcity into account, and ensures that the deer population do not expand beyond its carrying capacity \(k\). The importance of this feature is highlighted by a potential behavior observable in the first model: without mountain lions the deer population will expand infinitely. While California’s natural carrying capacity for deer is an unexpectedly contentious topic, the deer population peaked at approximately 2 million before declining substantially to its current level of 540,000.
Second Model Behavior

The phase plane of the second model is shown below. As the model is not linear, the Jacobian (matrix below) serves as a useful linear approximation.

Parameters:
\[ a = 0.875, \quad b = 1.8, \quad c = 0.2, \]
\[ d = 0.1, \quad f = 0.44, \quad k = 1000 \]

Equilibrium points: \((757, 44), (0,0), (1000, 0)\)

Jacobian:
\[
\begin{bmatrix}
2ax - by(y + 1) & -bx(fx + 1) \\
\frac{dy(y + 1)}{(fx + y + 1)^2} & -c + \frac{dx(fx + 1)}{(fx + y + 1)^2}
\end{bmatrix}
\]

Using this Jacobian, the equilibrium points can be categorized. The Jacobian for the only equilibrium where both species survive is shown below, along with the useful indicators it provides.

\[ J = \begin{bmatrix}
-0.45896 & -3.1775 \\
0.0014 & -0.2347
\end{bmatrix}, \quad \lambda_1 = -0.257, \quad \lambda_2 = -0.437, \quad tr(J) = -0.694, \quad det(J) = 0.112
\]

This makes \((757, 44)\) a nodal sink (stable). Using the same techniques, \((0,0)\) and \((1000, 0)\) can both be categorized as saddle points.

When initial populations are non-zero, they will eventually sink into a single equilibrium of 757 deer and 44 mountain lions. Unlike the previous model, the equilibrium is always approached (nodal sink) rather than circled around (center).
To what extent do you agree with the following statement: “I feel that using Slopes increased my understanding of the mathematical models in my project.”

- 37.5% Strongly agree
- 58.3% Agree
- Neither agree nor disagree
- Somewhat disagree
- Strongly disagree
- I did not use Slopes for this project
“It's good just because visualizing helps a lot to be able to understand, especially when you get to higher levels of math and things get kind of hard to understand sometimes.”

“I really love how interactive it is ... You can move it around and manipulate it. I like being able to click to see, okay, what does the solution with this initial condition look like?”
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Questions?

Check out the apps at
http://www.slopesapp.com
http://www.wavespdeapp.com