SIMIODE EXPO: SCUDEM V Birds and Bicycles

Team 112 Miles Pophal, Chenming Zhen, Henry Bae, Coach: Anthony Stefan

Melbourne, FL (USA) 32901
SIMIODE
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Introduction: Outline

We seek to

- Explain the problem clearly and concisely
- Apply natural assumptions
- Show how the general model follows from those assumptions
- Reduce the model to an idealized case for sanity checking
- Demonstrate the model with some simulations
- Conclude with a discussion on initial conditions
Problem: Problem Statement

A viral video shows a bird perched on a bicycle wheel able to move itself so the wheel spins, our task is to model the phenomena with a small apparatus attached to a wheel able to move a mass to generate the motion.
Problem: Our Apparatus

Our apparatus would

- Consist of a small piston capable of moving a mass radially outward and inward
- May be imparted with initial angular displacement or velocity from some initial lateral movement
- Comes from the bird leaning to start the motion
- Satisfies initial conditions to be used later of initial angular position and angular velocity
Problem: Simulation Example
Problem: Equations of Motion

From physics we know:

- Linear velocity \( v \):
  \[
  v = r(\theta)\dot{\theta}
  \]

- Angular position (recall measured from the vertical) just \( \theta(t) \)

- Coordinate of center of mass of apparatus
  \[
  x(\theta) = r(\theta)\sin(\theta), \quad y(\theta) = r(\theta)\cos(\theta)
  \]

- Here \( r(\theta) \) is the distance from the axis of rotation (the axle) to the center of mass of the whole system.
Problem: Assumptions

- Our bicycle wheel and apparatus represents a rigid body
- Our apparatus impacts motion by moving close or away from the bicycle wheel
- When we think rotational motion the analogy of Newton’s second law becomes:

\[
\sum \tau_i = l \alpha \iff \sum \tau_i - l \alpha = 0
\]

Where \( \tau_i \) represent torques and \( l \) is the moment of inertia, and \( \alpha \) is the angular acceleration \( \ddot{\theta} \) wrt time \( t \). And the 0 represents the zero vector since it arises from a cross product.
- The time our piston moves between states has negligible impact on the motion
Model Creation: Free Body Diagram

\[ F_G + F_{air} \]

\[ F_f \]

\[ F_{air} \]

\[ F_G \]

\[ \theta \]

\[ r \]
Following the previous slide we provide some detail to the torque equation

- So our apparatus generates a torque by decreasing or increasing its distance from the axis of rotation (center of wheel)
- The principal forces which generate torque are the apparatus (by gravity), air resistance, and friction
- These forces act on close enough points of contact to be treated the same
- When forces are tangential their torque component will be (in terms of the radius $r$)

$$\tau = r \times F \implies |\tau| = |F||r|\sin(\theta) = |F||r|$$
Reducing the complexity

- Our main equation

\[ \sum |\tau_i|\text{sgn}(\tau_i) - l|\alpha| = 0 \]

The \text{sgn} term represents the direction of the torque

- Torque by air resistance is proportional to velocity squared

\[ \tau_{\text{air}} = rF = rk(v)^2 = kr(r\dot{\theta})^2 = kr^3(\dot{\theta})^2 \]

- Torque by friction is proportional to normal force

\[ \tau_{\text{friction}} = \mu rF = \mu mv^2 = m\mu r^2(\dot{\theta})^2 \]

- Torque by gravity is standard

\[ \tau_g = mgr \sin(\theta) \]
**Model Creation: Equation**

- Moment of inertia term for a uniform wheel

\[ I(\ddot{\theta}) = mr^2(\theta)\ddot{\theta} \]

- Combining all terms to get

\[-mr^2(\theta)\frac{d^2\theta}{dt^2} - (kr^3(\theta) + m\mu r^2(\theta)) \left(\frac{d\theta}{dt}\right)^2 + mgr(\theta)\sin(\theta) = 0\]

- Since \( r(\theta) \neq 0 \) our equation reduces to

\[-mr(\theta)\frac{d^2\theta}{dt^2} - (kr^2(\theta) + m\mu r(\theta)) \left(\frac{d\theta}{dt}\right)^2 + mg \sin(\theta) = 0\]
Defining \( r(\theta) \)

\[
r(\theta) = r_1 + r_2 H(\sin(\theta))
\]

- \( H \) — Heaviside step function

\[
r(\theta) = r_1 + r_2 H(\sin(\theta)) = \begin{cases} 
  r_1 + r_2, & \sin(\theta) \in (0, 1] \\
  r_1, & \sin(\theta) \in [-1, 0]
\end{cases}
\]
Model Creation: Ideal Case

Reduction to ideal case by

- Lubricate axle eliminates frictional torque
- Apparatus is quite small eliminates air resistance
- With no dampening the velocity should increase without bound

\[-mr(\theta) \frac{d^2 \theta}{dt^2} + mg \sin(\theta) = 0\]

Alternatively:

\[r(\theta) \frac{d^2 \theta}{dt^2} = g \sin(\theta)\]
For clarity we used $t \in [0, 2]$ s, and some approximate initial conditions:

- $r_1 = 0.7$ m
- $r_2 = 0.1$ m
- $m = 5$ kg
- $g = 9.8$ m/s$^2$
- $k = 0.001$ kg m$^{-1}$
- $\mu = .01$
- $\theta(0) = 0$ rad
- $\dot{\theta}(0) = 4$ rad s$^{-1}$

Conditions can be changed from measurements as these were just estimated.

Applied a fourth order Runge-Kutta scheme for second order ODEs because of the incredible variance from initial conditions.

A longer time frame will be shown at the end of the analysis for further review.
The models overlayed look like:

![Ideal and Full Models Angle vs. Time](image1)

![Cos(Angle) vs. Time Ideal and Full Models](image2)
Running $t$ from $t \in [0, 4]$ interval

![Graph of Angle vs. Time Longer Time](image1)

![Graph of cos(Angle) vs. Time Longer Time](image2)
● Real case might lag slightly behind the ideal case and may even begin to fall back

● This provides a sanity check since air resistance and friction go against the motion of the apparatus

● The model is very difficult to predict and is very sensitive to changes in initial conditions

● Predictions for longer range times become more inaccurate meaning the longer time difference from our numeric scheme
Maximal Speed: Critical Value

From calculus we know a maximum value for speed can occur either on the boundary of your domain or at a value where the derivative is 0, so setting $\ddot{\theta} = 0$ we can get:

$$-(kr^2(\theta) + m\mu r(\theta)) \left(\frac{d\theta}{dt}\right)^2 + mg \sin(\theta) = 0$$

Hence our angular velocity becomes

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{mg \sin(\theta)}{kr^2(\theta) + m\mu r(\theta)}$$

$$\frac{d\theta}{dt} = \sqrt{\frac{mg \sin(\theta)}{kr^2(\theta) + m\mu r(\theta)}}$$
Maximal Speed: Critical Value

This only gives the maximum angular velocity so get the linear velocity we need a factor of radius meaning:

\[ v = r(\theta) \frac{d\theta}{dt} = r(\theta) \sqrt{\frac{mg \sin(\theta)}{kr^2(\theta) + m\mu r(\theta)}} = \sqrt{\frac{mgr^2(\theta) \sin(\theta)}{kr^2(\theta) + m\mu r(\theta)}} \]

With some factoring:

\[ v = \sqrt{\frac{mg \sin(\theta)}{k + \frac{m\mu}{r(\theta)}}} \]
Maximal Speed: Final Expression and Boundary

Maximized when \( \sin(\theta) = 1 \) evaluating to:

\[
v_{\text{max}} = \sqrt{\frac{mg}{k + \frac{m\mu}{r_1+r_2}}} = 16.8575 \text{ m/s}
\]

Note about the boundary and value

- Initial conditions change the left endpoint so nothing definite to say
- Unbounded right side so speed could increase without bound
- This max velocity assumes it actually occurs, but if we speed up without bound this quantity is obviously false and moreover nonexistent
Can we still get motion if we start at the bottom? Of course with a high enough initial kick ($\dot{\theta}(0) = 8 \text{ rad s}^{-1}$), here’s what motion would look like in the ideal and full models:
Initial Conditions: No initial velocity

With no initial velocity we can see no motion takes place, but if we change the position slightly we can still recover motion:

This also passes a sanity check since with no motion we don’t turn and with no initial velocity it takes longer to get moving so we move less in a single time interval.
Initial Conditions: Limitations

- At our speeds air resistance and rotational friction are nearly meaningless.
- At longer intervals our numerical estimates become more unreliable (RK-4).
- We rely on knowing the center of gravity for the apparatus.
- Heaviside function is not a perfect fit only a simplification.
To summarize:

- Set out to model the motion of a bird on a bicycle wheel
- Found with a given radius function we can keep spinning with increasing speed
- Depending on initial conditions we may not be able to even complete a rotation (e.g. lacking rotational velocity)
- Found an expression for the maximum speed
- Sanity checked our model with some numerical simulations
- Considered the limitations of our model
Conclusion: Our SIMIODE Experience

Working together with everyone

The epiphany moment

Applications in modern science
Conclusion: Thank You

A special thank you to Anthony Stefan for getting us interested and to SIMIODE/SCUDEM for taking the time to watch, review, and judge our presentation!