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STUDENT VERSION
Home Heating

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STATEMENT

Suppose you're going away on vacation over Christmas, for a total of four days (96 hours). When you leave the house the inside temperature is a comfortable 21°C; you want it at that same temperature when you return. Since it's winter, this means you'll have to run the furnace. To minimize the costs/fuel consumption, should you

1. Just leave the furnace on (thermostat set to 21°C the whole time you're gone), or should you
2. Set the thermostat so the furnace stays off for most of the four days, but comes on just in time to heat the house back to 21°C when you return at $t = 96$ hours?

Scenario 2 assumes there's no problem with letting the house get cold, e.g., burst pipes, tropical plants, small pets, etc. It also assumes you have a suitably programmable thermostat.

In summary, the central question of interest is

Is it more economical to keep the house warm, or to reheat it?

Newton's Law of Cooling

Newton's Law of Cooling states that if an object with temperature $y(t)$ (here t is time, y temperature) sits in an environment with ambient temperature A then $y(t)$ obeys, to good approximation,

$$\frac{dy}{dt} = -k(y(t) - A) \tag{1}$$

where $k > 0$ is a constant. The ambient temperature A might be a function of time. There are many assumptions implicit in the model, among them that the object has a single well-defined temperature throughout. Thus in what follows we'll treat the object as being "negligible" in size, even though it's a house!

Exercise 1: Solve (1) with initial condition $y(0) = y_0$; assume A is constant. What is the long-term behavior of the solution? How does it depend on k?

Exercise 2: Suppose we measure time t in hours and $y(t)$ denotes the temperature of the house in degrees Celsius. Take $t = 0$ as the time you leave the house, $y(0) = 21$, $k = 0.01$, and suppose $A = 0$. What are the units on k? Use the solution from Exercise 1 to determine how cold the house will be when you return at time $t = 96$ hours. Pretty cold, huh?

Modifying Newton's Law of Cooling for a Heat Source

However, the house has a furnace, so in our model we can suppose that heat energy is being actively generated at some rate. Let $y(t)$ be the (inside) temperature of the house and A the ambient (outside) temperature. In this case Newton's Law of Cooling (1) can be modified as

$$y'(t) = -k(y(t) - A) + r(t) \tag{2}$$

where $r(t)$ is a function that is proportional to the rate of energy generation, which we allow to be a function of time. (If you want to be precise, in the simple case that the object has mass M, specific heat c, and is of negligible extent, $r(t) = \frac{E(t)}{Mc}$, where $E(t)$ is the rate of energy generation).

In what follows we will assume temperature is measured in degrees Celsius and, given the scale of the problem, time is measured in hours. The function $r(t)$ itself then has units of degrees per hour. One might think of $r(t)$ as follows: if at some instant $t = t_0$ we have $y(t_0) = A$ (that is, the house is at ambient temperature) and the furnace is on, then from (2) we have $y'(t_0) = r(t_0)$, so $r(t_0)$ quantifies the rate at which the furnace is raising the temperature of the house at that instant.

Analysis of Scenario 1: Keeping The House Warm

Suppose the house temperature $y(t)$ obeys (2); $r(t)$ would be proportional to the rate at which the furnace consumes energy. Again throughout the analysis suppose, for simplicity, that $A = 0°$C and $k = 0.01$ (based on watching the author's house cool on a cold winter day when the power went out).

In Exercise 2 above you of course found that the house is quite cold if you leave the heat off for four days. But you want the house to be at 21°C, the same as when you left. How much energy will be needed to heat the house if you leave the furnace set to maintain 21°C the whole time?

Exercise 3: Suppose $r(t) = r_0$ in (2), where r_0 is a constant chosen so as to maintain a constant 21°C in the house—what constant r_0 is required? Hint: choose r_0 so that $y(t) = 21$ is an equilibrium solution to (2).

Exercise 4: Draw a phase portrait for (2) assuming that $r(t) = r_0$ is constant. What does an equilibrium represent?

Exercise 5: Note that $r(t)$ is directly proportional to the power output of the furnace, and so $\int_0^{96} r(t)\,dt$ is directly proportional to the total energy expended by the furnace during the four days, which itself should be proportional to the cost of running the furnace for that time period. Compute this integral (hint: this is trivial). The result is a measure of the cost of keeping the house at 21°C.

A note for the picky: you can't run a typical furnace at any power you want—it's either on or off. So you can't choose any r_0 you want, but you can fake it: if a given choice for r_0 corresponds to, say, 50 percent of the furnace maximum output, you could run the furnace full on for 30 minutes, then off for 30 minutes, then on for 30, etc., i.e., run it at a 50 percent duty cycle. Over a four day time span it would look pretty much like the furnace running at 50 percent.

Analysis of Scenario 2: Reheat The House

Here's an alternate strategy for making sure the house is 21°C when you return: Turn the heat off when you leave, but program the thermostat to turn on the furnace full blast at some time $t = t_1$, with $t_1 < 96$, to reheat the house to 21°C at exactly time $t = 96$ when you return home. The value of t_1 should be chosen as large as possible (i.e., turn the furnace on at the last possible minute). Of course this assumes, somewhat unrealistically, that you know the outside ambient temperature at all times, but it's a start.

Will this be more or less efficient than the previous strategy? Answer the following questions to find out!

Exercise 6: Suppose the furnace is off from time $t = 0$ to some unspecified time $t = t_1 < 96$. Let's use $y_1(t)$ to denote the temperature of the house during this time period, so $y_1(t)$ obeys (1) with $y_1(0) = 21$. Use the results of Exercise 1 to write out $y_1(t)$ explicitly when $k = 0.01$ and $A = 0$.

Exercise 7: At some time $t = t_1$ we need to turn the furnace on "full blast" to reheat the house. Assume that in this case the function $r(t) = 3$ degrees per hour (meaning the furnace is capable of heating the house at a max rate of 3 degrees per hour, at least when $y(t) = 0$). From time $t = t_1$ to time $t = 96$ the temperature of the house will be given by a function $y_2(t)$ that satisfies the DE (2) with the *final condition* $y_2(96) = 21$. Solve (2) with $y_2(96) = 21$ to find $y_2(t)$.

Could we simply have modeled $y_2(t) = 3t + b$, where b is chosen so that $y_2(96) = 21$? Explain.

Exercise 8: Refer to Figure 1. The function $y_1(t)$ gives the temperature of the house for $0 \le t \le t_1$, while $y_2(t)$ gives the temperature for $t_1 \le t \le 96$. The value of t_1 when the furnace should be turned on is dictated the condition $y_1(t_1) = y_2(t_1)$ (think: why is this appropriate?) Use this observation to find t_1.

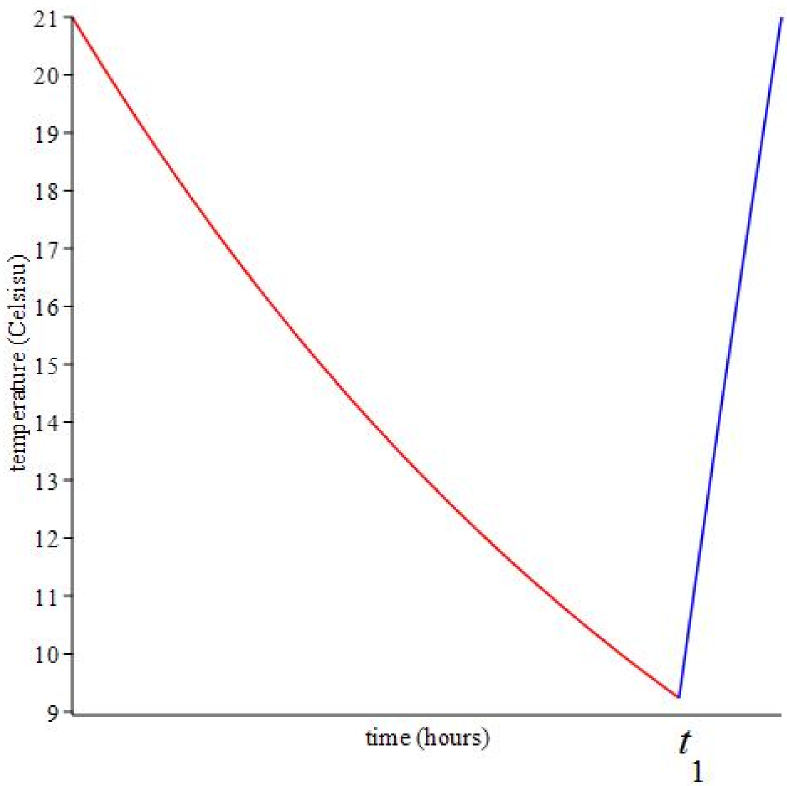

Figure 1. Function $y_1(t)$ (red), $y_2(t)$ (blue) for $0 \le t \le 96$.

If the furnace does not shut off, will the temperature in the house continue to rise without bound after 96 hours?

Exercise 9: As in Scenario 1, a measure of how much it costs to heat the house is given by $\int_0^{96} r(t)\,dt$. What is the integral in this case? (Note, $r(t) \equiv 0$ for $t < t_1$, so you really only need to worry about $t_1 \le t \le 96$).

Exercise 10: Compare the total "energy" usage from Exercise 5 to that from Exercise 9. Which is more economical, reheating the house or just leaving it at 21°C?

Exercise 11: (Harder!) You're going to be gone from time $t = 0$ to time $t = T$. The ambient temperature outside is A (constant). Let the initial temperature of the house be y_0, which also happens to be the temperature you want it when you return; assume $y_0 > A$. The cooling constant is simply k, and the furnace is capable of a maximum output of $r(t) = r_M$, where $r_M \geq k(y_0 - A)$ (this ensures that the furnace is at least capable of maintaining temperature y_0!)

Is there ANY combination of constants T, p_0, A, k, and r_M that make it more economical to just leave the furnace on while you're gone? Or can you prove that this strategy is never beneficial?