

STUDENT VERSION

Modeling Historic Battles Using Lanchester's Laws

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Abstract: Lanchester's laws are used to calculate the relative strengths of military forces. The Lanchester equations are differential equations describing the time dependence of two armies' strengths A and B as a function of time, with the function depending only on A and B . In this project, you will model modern combat using Lanchester's Laws.

Keywords: Lanchester's Laws, systems, numerical solutions

Tags: Sensitivity analysis, equilibrium, spreadsheets, MATLAB

STATEMENT

Lanchester's laws are used to calculate the relative strengths of military forces. The Lanchester equations are differential equations describing the time dependence of two armies' strengths A and B as a function of time, with the function depending only on A and B . During World War I, M. Osipov and Frederick Lanchester independently developed a series of differential equations to demonstrate the power relationships between opposing forces [2]. Among these are what is known as Lanchester's Linear Law (for ancient combat) and Lanchester's Square Law (for modern combat with long-range weapons such as firearms).

Lanchester's Linear Law

For ancient combat, between soldiers with spears, say, one soldier could only ever fight exactly one other soldier at a time. If each soldier kills, and is killed by, exactly one other, then the number of soldiers remaining at the end of the battle is simply the difference between the larger army and the smaller, assuming identical weapons. For example, consider a Red army with 42 soldiers and a Blue army with 30 soldiers. Assuming the same weapons were used for each side, the Red Army would win the battle with 12 soldiers remaining.

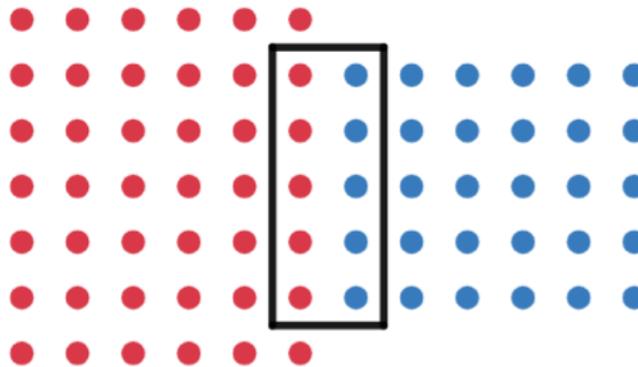


Figure 1: Representation of a Battle Between an Army of Size 42 and 30

The linear law also applies to unaimed fire into an enemy-occupied area. The rate of attrition depends on the density of the available targets in the target area as well as the number of weapons shooting. If two forces, occupying the same land area and using the same weapons, shoot randomly into the same target area, they will both suffer the same rate and number of casualties, until the smaller force is eventually eliminated: the greater probability of any one shot hitting the larger force is balanced by the greater number of shots directed at the smaller force.

Lanchester's Square Law

With firearms engaging each other directly with aimed shooting from a distance, both sides can attack multiple targets and can receive fire from multiple directions. The rate of attrition now depends only on the number of weapons shooting.

Lanchester determined that the power of such a force is proportional not to the number of units it has, but to the square of the number of units. This is known as Lanchester's square law.

More precisely, the law specifies the casualties a shooting force will inflict over a period of time, relative to those inflicted by the opposing force. In its basic form, the law is only useful to predict outcomes and casualties by attrition. It does not apply to whole armies, where tactical deployment means not all troops will be engaged all the time.

Suppose that two armies, Red and Blue, are engaging each other in combat. Red is shooting a continuous stream of bullets at Blue. Meanwhile, Blue is shooting a continuous stream of bullets at Red.

Let A represent the number of soldiers in the Red force. Each one has offensive firepower α , which is the number of enemy soldiers it can incapacitate per unit time. Likewise, Blue has B soldiers, each with offensive firepower β . Note that both A and B are functions of time, i.e. $A(t)$ and $B(t)$.

1. Write a system of differential equations that models the rate at which the Red and Blue soldiers are changing.
2. Solve this system of differential equations analytically. Explain why your answer makes sense in terms of combat. Does the solution agree with what is stated above about Lanchester's Square Law?
3. Although we can solve the system analytically, it will be easier to visualize the behavior of the system if we solve it numerically and graph the results. For a reasonable set of parameters, solve this system of differential equations (1) numerically using software of your choice (MATLAB, Excel, etc.) and include a graph showing the behavior of the population of Red and Blue soldiers over time.
4. Using the solution you generated in question 2, vary the parameters to determine the outcome in the following scenarios.
 - a. The two sides have equal firepower ($\alpha = \beta$).
 - b. The two sides have an equal number of soldiers ($A = B$).
 - c. If $A > B$ and $\alpha > \beta$.
 - d. If $A < B$ and $\alpha < \beta$.

5. What is the outcome if $A > B$ and $\alpha > \beta$ or $A < B$ and $\alpha < \beta$? Conduct a sensitivity analysis to determine what the winning side depends on. (Hint: reread the background information above on Lanchester's Square Law and verify that this holds true for various values of A , B , α , and β). What does this mean about the relative importance of firepower versus army size?
6. Revisiting your analytical solution to the system of differential equations, verify your empirical results from question 6 using the fact that $\frac{dB}{dA} = \frac{\alpha A}{\beta B}$.
7. Based on your results in questions 4 and 5, does this system appear to have any equilibrium values? Explain what this means in the context of combat.

Extension

Lanchester's Laws are only valid when each unit (soldier, ship, etc.) can kill only one equivalent unit at a time. For this reason, the law does not apply to machine guns, artillery, or nuclear weapons. The law requires an assumption that casualties accumulate over time: it does not work in situations in which opposing troops kill each other instantly, either by shooting simultaneously or by one side getting off the first shot and inflicting multiple casualties. The Salvo Combat Model extends the ideas of Lanchester's Laws to apply to anti-ship missile battles between modern warships [1]. Do some background research on the Salvo Combat Model and explain the similarities and differences to Lanchester's Laws.

Homework

Choose a historic battle from the Civil War (for example, the Battle of Gettysburg [3]) and model it using Lanchester's Laws. Determine the number of soldiers on each side at the beginning of combat and make reasonable assumptions about relative firepower. Does Lanchester's Square Law accurately capture the result of the battle? Alternatively, you could apply Lanchester's Law to a first-person shooter video game of your choice. Again, you will need to determine the starting number of soldiers and make reasonable assumptions about relative firepower. In either case, write a brief report explaining your results.

REFERENCES

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