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STUDENT VERSION

Bad Vibrations: Modeling a Building During an Earthquake Part I: No Damping

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SCENARIO DESCRIPTION

An engineer is responsible for designing something that functions properly, not only in the ideal situation, but also in nonideal situations that may occur. One cannot plan for every possibility, but likely scenarios should be taken into account. Vibrations occur in almost every object, not only in mass-spring systems. Vibrations are the reason a loud sound of the correct pitch can cause glass to shatter. The frequency of this pitch is referred to as the *resonant frequency* of the glass.

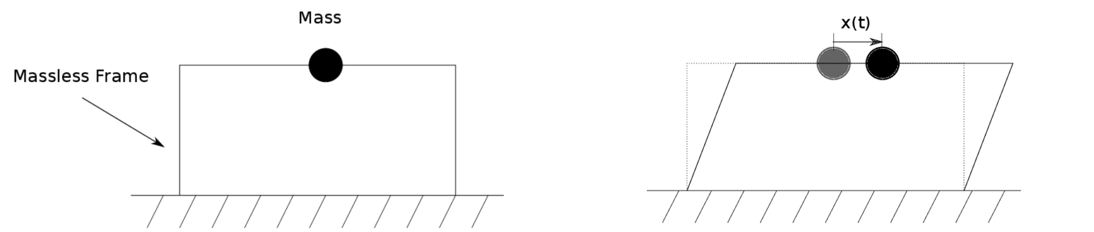
Vibrations are important to take into account when designing a space ship. Both the Saturn and Ares rocket experience large vibrations during the first stage when solid fuel was burned due to small disturbances in the flow of the fuel. If these vibrations are at or near the resonance frequency of the spacecraft, the result could be disastrous. During an unmanned Apollo 6 flight, large vibrations caused the main engine to shut down [5].

Ground vibrations need to be considered in many situations. Most of these vibrations are too small to be noticed by us (without a seismograph or similar device), but sometimes they are large enough to be felt. Buildings must be designed with these vibrations in mind, especially in places that are more likely to experience earthquakes. Our goal is to analyze the movement of a building during an earthquake. Engineers must ensure the movement is small enough that there will not be structural damage to the building or discomfort for the occupants due to motion of the structure.

In our analysis, we are going to consider *simple* structures that can be idealized as a single point mass. In particular, we are going to analyze the roof of a one-story building. Assuming that the roof is uniform, we can approximate the roof by a single point mass in the center of the building;

see Figure 1. The roof itself is very stiff and will resist motion. It is horizontal movement of the supporting column that will cause the roof to move.

Your goal is to find the resonance frequency of a building. Similar to a glass, this is the frequency that you do not want the ground to be vibrating at, as it will cause the building to move too far and (eventually) collapse or, in the case of glass, shatter.



(a) Considering the roof as a point mass at the center (b) Displacement of the roof is measured by $x(t)$

Figure 1: Motion of the center of mass of the roof of a one-story building

ACTIVITY 1 – NO VIBRATIONS

First we consider an (idealized) one-story building, while ignoring damping effects. We look at the simplified scenario where the roof is supported by a single supporting column. The displacement of the center of mass of the roof, $x(t)$ (see Figure 1), can be modeled as

$$mx'' + kx = 0, \qquad x(0) = x_0, \;\; x'(0) = x_1 \tag{1}$$

where m is the total mass of the roof and k is the stiffness of the supporting column. It is known that

$$k \text{ is proportional to } EI/h^3$$

where E is the elastic modulus of the supporting column, I is the moment of inertia of the column about the bending axis, and h is the length of the column [1].

Problem 1. What units should the constant k have for the differential equation (1) to make sense? *Note: you will need to be general here (for example "time") since explicit measurements (for example "seconds") were not given yet.*

Problem 2. If any type of force (such as an earthquake) acts on the system at $t = 0$, the initial values would be nonzero. Find the (analytic) solution for $x(t)$ in this case. Explain why this solution does or does not make sense physically (in the long term).

Problem 3. Suppose that the mass of the roof is increased (for example, by installing a different type of roof or modeling a larger roof/building). Is the displacement more or less (or the same) in this case? Justify your conclusion.

Problem 4. Suppose that the height of the building (and therefore the length of the supporting column) is increased. Is the displacement more or less (or the same) in this case? Justify your conclusion.

Energy

Let $x(t)$ be the displacement of a mass (of size m). The **kinetic energy** of the mass (at time t) is given by

$$E_k(t) = \frac{\text{mass} \times \text{velocity}^2}{2} = \frac{1}{2}mv^2(t) = \frac{1}{2}m[x'(t)]^2.$$

The **potential energy** of a mass m (at time t) is defined as

$$E_p(t) = \frac{\text{stiffness} \times \text{position}^2}{2} = \frac{1}{2}kx^2(t).$$

The **total energy** of the mass (at time t) is

$$E(t) = E_k(t) + E_p(t).$$

Problem 5. Show that, in this case, the total energy of the system is conserved; in other words, the energy is constant.

ACTIVITY 2 – VIBRATIONS

If an earthquake occurs, the ground will be displaced some distance $x_G(t)$. Then that total lateral deflection of the roof is

$$x_T(t) = x(t) + x_G(t)$$

where $x(t)$ is the displacement caused by vibrations of the building and $x_G(t)$ is the movement caused by the movement of the ground (and therefore, supporting column); see Figure 2.

Problem 6. Explain why, during an earthquake, it is (physically) reasonable to assume that

$$x''_G(t) = -\frac{A}{m}\sin(\omega t).$$

Even though this is reasonable, explain why this assumption/model is not ideal.

Problem 7. Assuming that there is no damping, we have

$$mx''_T + kx = 0. \tag{2}$$

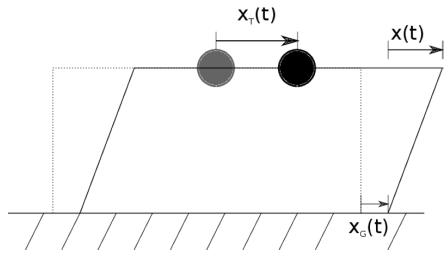


Figure 2: Displacement of a roof caused by an earthquake.

Both the ground and the roof/building are accelerating. However, only the roof (not the ground) is affected by the stiffness of the supporting column. Find a differential equation for $x(t)$ (in terms of $x_G(t)$).

Problem 8. Using the function $x_G(t)$ above (and assuming that the building was not initially moving), solve the associated initial-value problem for $x(t)$.

Problem 9. The roof will collapse if it moves too much. Find one value of ω that will definitely cause too much motion (in the long term) and cause the building to collapse (eventually). Describe physically and mathematically what is occurring in this case. *Hint: Think of resonance and carefully consider all possible solutions to Problem 8.*

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