Scientia Iranica D (2012) 19 (3), 733–738

Sharif University of Technology
Scientia Iranica
Transactions D: Computer Science & Engineering and Electrical Engineering
www.sciencedirect.com

# Complex dynamic behaviors of the complex Lorenz system

**M. Moghtadaei, M.R. Hashemi Golpayegani** *

*Complex Systems and Cybernetic Control Lab., Faculty of Biomedical Engineering, Amirkabir University of Technology, Tehran, P.O. Box 1591634311, Iran*

Received 4 August 2010; revised 11 September 2010; accepted 29 November 2010

**KEYWORDS**
Lorenz system;
Complex variables;
Chaos;
Hyper-chaos.

**Abstract** This study compares the dynamic behaviors of the Lorenz system with complex variables to that of the standard Lorenz system involving real variables. Different methodologies, including the Lyapunov Exponents spectrum, the bifurcation diagram, the first return map to the Poincaré section and topological entropy, were used to investigate and compare the behaviors of these two systems. The results show that expressing the Lorenz system in terms of complex variables leads to more distinguished behaviors, which could not be achieved in the Lorenz system with real variables, such as quasi-periodic and hyper-chaotic behaviors.



## 1. Introduction

In 1963, the concept of chaos was introduced by Edward Lorenz via his 3D autonomous system [1]. Over time, interest in the complex dynamic behaviors of nonlinear systems increased, due to their potential applications in different fields, such as detecting changes of biological signals (mostly EEG) in different abnormalities [2], data and image encryption [3], studying sunspot cycles [4], and lasers [5]. Consequently, a large number of novel systems have been developed based on the original Lorenz system [6–9].

A more complex form of dynamic behavior, hyper-chaos, was introduced by Rössler in 1979, as a higher form of chaos, with at least two directions of hyperbolic instability on the attractor [10]. In contrast to chaotic attractors, the hyper-chaotic attractor has attracted more attention owing to its higher complexity, which makes it harder to predict. Since messages masked by simple chaotic systems are not always safe, the use of higher dimensional hyper-chaotic systems to secure communication, and data and image encryption has been suggested because of their higher unpredictability [11–13]. By definition [10], complex hyper-chaotic behavior cannot be observed in 3-D systems. So, in order to achieve hyper-chaos, one should always add at least one extra variable to the three variable Lorenz equations [6,8,14–18].

Choosing different feedback controllers to the Lorenz equations can lead to different chaotic and hyper-chaotic systems. Many researchers [6,8,9,14–21] have introduced their hyper-chaotic systems by adding extra variables to the Lorenz equations. In contrast, in this paper, we show that, indeed, simply changing Lorenz variables to complex forms not only keeps the chaotic dynamics of the Lorenz equations but will also change the system into a higher dimensional system, where it has the potential to be hyper-chaotic for some ranges of parameter value. There are numerous ways to evaluate chaotic behavior in systems, such as estimation of the fractal dimension [22–24], correlation dimension [24], Lyapunov Exponent spectrum [24,25], and bifurcation diagram [24,26] etc. Here, we investigate the level of chaos in our system through numerical simulations by means of computing the Lyapunov Exponents spectrum, bifurcation diagram, first return map to the Poincaré section, topological entropy of the first return map, and the phase diagram.

In the following sections, first, the two Lorenz system forms that are based on real and complex variables are described, and then the requirements of chaotic and hyper-chaotic behaviors, and our methods for measuring them are briefly explained. Subsequently, the results of our analysis on the behavior of these two forms of Lorenz system are reported and finally in the conclusion and discussion section, the main distinguished dynamical and behavioral features of the complex form of the Lorenz system are compared with those of its standard real form.

* Corresponding author. Tel.: +98 21 64542389.
*E-mail addresses:* mmoghtadaei@aut.ac.ir (M. Moghtadaei), mreza.Hashemig@gmail.com (M.R. Hashemi Golpayegani).
Peer review under responsibility of Sharif University of Technology.

ELSEVIER Production and hosting by Elsevier

1026-3098 
doi:10.1016/j.scient.2010.11.001

## 2. Complex and real Lorenz systems, and methods to quantify their behaviors

The Lorenz system is described as [1]:

$$f:\begin{cases}\dot{x}=a(y-x)\\ \dot{y}=cx-y-xz\\ \dot{z}=xy-bz.\end{cases} \tag{1}$$

Some important basic features of this system are:

1. It is autonomous, which means that time does not explicitly appear on the right hand side.
2. The equations involve only first order time derivatives, so the evolution depends only on the values of $x$, $y$, and $z$ at the time.
3. Due to the terms of $xz$ and $xy$ in the second and third equations, the system is non-linear.
4. The system is dissipative when the following inequality holds:

   $$\nabla f=\frac{\partial\dot{x}}{\partial x}+\frac{\partial\dot{y}}{\partial y}+\frac{\partial\dot{z}}{\partial z}=-a-1-b<0.$$

   Since parameters $a$ and $b$, denoting the physical characteristics of the air flow, are positive, the inequality always holds and, thus, solutions are bounded.
5. The system is symmetric, with respect to the $z$ axis, which means it is invariant for the coordinate transformation:

   $(x,y,z)\rightarrow(-x,-y,z)$

The dual form of the Lorenz system expressed in terms of complex variables is described by a similar formula [7]:

$$\begin{cases}\dot{x}=a(y-x)\\ \dot{y}=cx-y-x^{*}z\\ \dot{z}=xy-bz\end{cases} \tag{2}$$

where $x$, $y$, and $z$ are complex variables defined as $x=x_1+ix_2$, $y=y_1+iy_2$, $z=z_1+iz_2$.

The basic features of this complex form of Lorenz system are similar to the aforementioned features. That means it is autonomous, non-linear, dissipative with bounded solutions, symmetric with respect to the $z$ axis, and involves only first order time derivatives. Besides, this complex form is equal to the third order Lorenz when the imaginary part of all variables is equal to zero.

Chaos is described as the irregular, unpredictable behavior of deterministic, non-linear dynamical systems. In order to have chaotic behavior, simultaneous stretching and folding in the dynamics of the system are essential [27,28]. Stretching ensures the sensitivity to initial conditions and folding guarantees that the attractor is bounded [29]. Stretching and folding are equivalent to positive and negative Lyapunov Exponents, respectively. Therefore, in chaotic dynamics, stretching in one direction is observed by the existence of one positive Lyapunov Exponent in that direction [30]. For instance, in the three dimensional Lorenz system, certain values of parameters lead to a chaotic system with a hyperbolic instability in one direction; this chaotic behavior shows itself by one positive Lyapunov Exponent.

Hyper-chaos is a more complex form of chaos. In order to achieve hyper-chaos, the important requirements are as follows [6]:

1. The system should have a dissipative structure.
2. The smallest dimension of the system should be at least four.
3. The number of terms giving rise to instability should be more than one, which means the system should expand in at least two directions, at least one of which should have a nonlinear function.

As mentioned above, Requirements 1 and 2 are already satisfied for the Lorenz system involving complex variables. To satisfy the third requirement, the existence of at least two positive Lyapunov Exponents is necessary [14]. Therefore, in order to prove hyper-chaotic behavior in the complex form of the Lorenz system, we evaluated the presence of these two positive Lyapunov Exponents.

Using the method reported by Wolf et al. [25], the corresponding Lyapunov exponent spectrums of the real and complex Lorenz systems were computed by fixing two parameters, $a=10$ and $b=8/3$, and varying the third parameter, $c$, as the control parameter. The initial conditions in all these simulations are the same and are equal to $\left[x_{10},x_{2_0},y_{1_0},y_{2_0},z_{1_0},z_{2_0}\right]=[0.1,0.2,0.3,0.4,1,2]$.

In addition, the bifurcation diagram was plotted for our systems, which is the most useful graphical representation of the sequence of bifurcations that take place in the system when the control parameter changes [26]. Also, the first return map to the Poincaré section and the topological entropy of the first return map were computed to check the complexity in dynamics for both complex and real based Lorenz systems.

The first return map to the Poincaré section is a common tool for analyzing the existence and stability of periodic trajectories of dynamical systems. It is defined on a hyper-surface formed by a Poincaré section, which is transverse to the trajectory of the system [31]. In this study, these maps were formed by plotting the local maxima of the variable, $z$ (in the Lorenz system with real variables), and the real part of the variable, $z$ (in the Lorenz system with complex variables), against their next local maxima (at $c=30$).

The topological entropy quantifies the degree of complexity of a trajectory and it is the most important quantity related to the orbit growth [32]. We used the symbolic dynamics of the first return maps at $c=30$ to compute topological entropies by visualizing the complexity of the first return maps in the space of symbol sequences [33]. Comparing the topological entropy of both systems provides a finer distinction between the states of their complexity.

## 3. Dynamical analysis of the Lorenz system involving complex variables

The corresponding Lyapunov Exponent spectrums of the System (1) (real form of Lorenz equations), and System (2) (complex form of Lorenz equations), are shown in Figures 1 and 2, respectively. Also, the bifurcation diagrams of the state variables, $z$, in System (1), and $z_1$ in Systems (2) are shown in Figures 3 and 4, respectively. Considering Figures 1–4, one can see that the bifurcation diagrams of both systems are in complete agreement with their corresponding Lyapunov Exponent spectrums.

Figures 1 and 3 demonstrate that System (1) is chaotic for some specific ranges of the parameter, $c$. Regarding the complex form of Lorenz, based on Figures 2 and 4, it is obvious that this system is chaotic for small ranges of parameter $c$, and it is hyper-chaotic for a very wide range of parameter $c$. The numerical analysis shows that this complex form of Lorenz could evolve to quasi-periodic orbits as well.

Based on Figure 1, some critical values of parameter $c$, and their corresponding Lyapunov Exponents are listed in Table 1

Table 1: Some critical values of parameter $c$ and their Lyapunov exponents for System (1).

| $c$ | 1 | 23.9 | 69.8 | 71.5 | 92.2 | 100 |
|---|---|---|---|---|---|---|
| $\lambda_1$ | −0.00907 | 0.715651 | 0.236959 | 0.030498 | 0.009054 | 0.013227 |
| $\lambda_2$ | −2.66223 | 0.000308 | −0.32936 | −0.61033 | −0.17723 | −0.03342 |
| $\lambda_3$ | −10.9789 | −14.3071 | −13.5473 | −13.0569 | −13.4799 | −13.6265 |

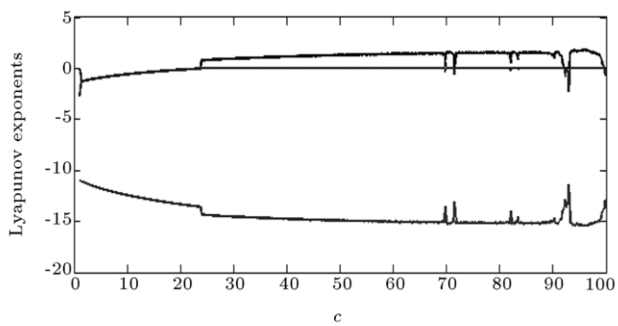


Figure 1: Lyapunov exponents spectrum of the Lorenz system involving real variables.

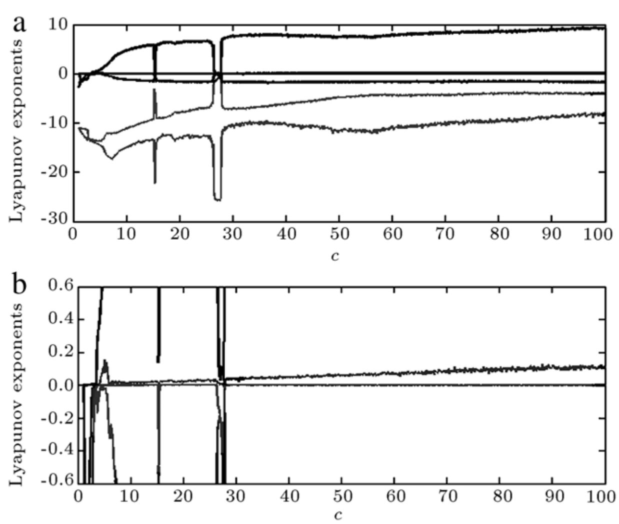


Figure 2: Lyapunov exponents spectrum of the Lorenz system expressed in terms of complex variables. (a) Auto-scaled; and (b) magnified.

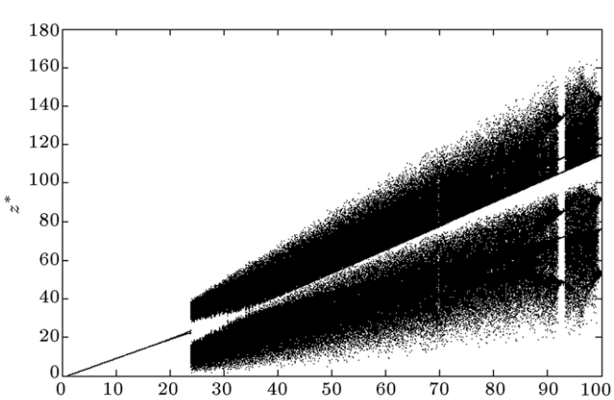


Figure 3: Bifurcation diagram of the Lorenz system involving real variables.

for System (1). The dynamics of the system are visualized accurately in its bifurcation diagram, shown in Figure 3; the

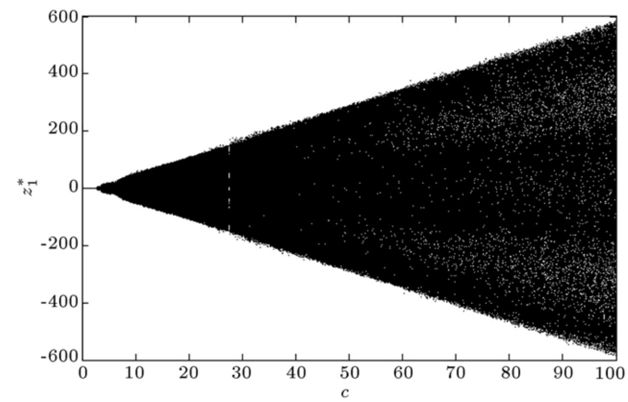


Figure 4: Bifurcation diagram of the Lorenz system involving complex variables.

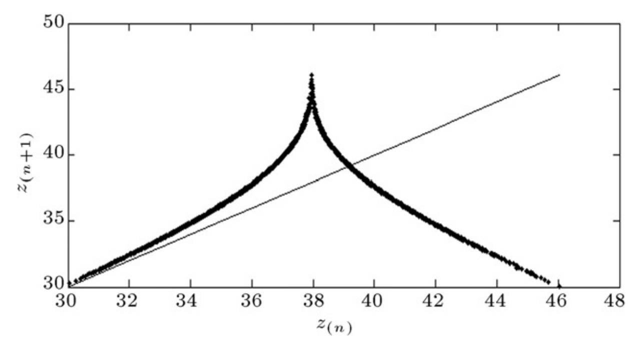


Figure 5: First return map of the Lorenz system with real variables for $c = 30$, when the Poincaré points are the local maxima of variable $z$.

critical points are the points at which there is a sudden change in the behavior of the system.

In the same way, the critical parameter values for the System (2), and their corresponding Lyapunov Exponents are listed in Table 2. One should note that these values are different from the critical parameter values of System (1).

Considering Figures 1 and 2, and Tables 1 and 2, we see that the dynamics of the two systems change by increasing parameter $c$. Besides, Systems (1) and (2) have different dynamical behaviors for different ranges of parameter $c$. This difference is summarized in Table 3.

When $c = 30$, the behavior of System (1) is chaotic, and the behavior of System (2) is hyperchaotic. The first return maps of both systems for this parameter value are shown in Figures 5 and 6. When visualized, as shown in Figure 5, for the Lorenz system with real variables, the map is unimodal with two monotonic branches split by a cusp. The point that separates the inreasing branch and the decreasing branch is a non-differentiable maximum point.

In contrast, the Poincaré first return map is not unimodal for the Lorenz system with complex variables, as shown in Figure 6.

These maps, specially the one for System (2), are pretty complicated. The convergent approximations of the entropy for both maps were computed and are shown in Figures 7 and 8.

Table 2: Some critical values of parameter $c$ and their Lyapunov exponents for System (2).

| $c$ | 1 | 3 | 3.5 | 5.1 | 6 | 15.4 | 27.5 | 27.8 | 50 | 100 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda_1$ | −0.00451 | 0.004348 | 0.002791 | 0.795696 | 1.558108 | 0.221139 | 0.093971 | 0.500816 | 7.507858 | 9.141735 |
| $\lambda_2$ | −0.00834 | −0.14935 | 0.001162 | 0.153863 | 0.005425 | 0.005762 | 0.007655 | −0.02163 | 0.062105 | 0.099071 |
| $\lambda_3$ | −2.66177 | −0.15614 | −0.03026 | −0.00341 | −0.00118 | −0.43745 | −0.31439 | −0.05344 | −0.00125 | 0.000136 |
| $\lambda_4$ | −2.66288 | −0.28416 | −0.03884 | −0.02571 | −0.32573 | −0.49748 | −0.6361 | −0.16921 | −1.67095 | −1.72657 |
| $\lambda_5$ | −10.9796 | −13.35 | −13.6179 | −13.6495 | −12.5572 | −4.13576 | −0.69013 | −1.65545 | −4.84548 | −3.98444 |
| $\lambda_6$ | −10.9832 | −13.3709 | −13.6277 | −14.5654 | −15.9937 | −22.1385 | −25.6568 | −24.9473 | −11.5099 | −8.36789 |

Table 3: Different behaviors of Systems (1) and (2) in different ranges of parameter $c$, and a comparison between the two systems.

| $c$ | 1–3.1 | 3.1–3.5 | 3.6–4.2 | 4.3–23.9 | 23.9–24.7 |
|---|---|---|---|---|---|
| Behavior of System (1) | Fixed point | Fixed point | Fixed point | Fixed point | Transient chaos (one positive Lyapunov exponent) |
| Behavior of System (2) | Periodic | Quasi-periodic | Chaotic | Hyper-chaotic | Hyper-chaotic |
| $c$ | 24.7–26.7 | 26.8–27.5 | 27.6–27.8 | 27.9–99.6 | 99.6–100 |
| Behavior of System (1) | One positive Lyapunov exponent | One positive Lyapunov exponent | One positive Lyapunov exponent | Chaotic | Periodic |
| Behavior of System (2) | Hyper-chaotic | Chaotic | Periodic | Hyper-chaotic | Hyper-chaotic |

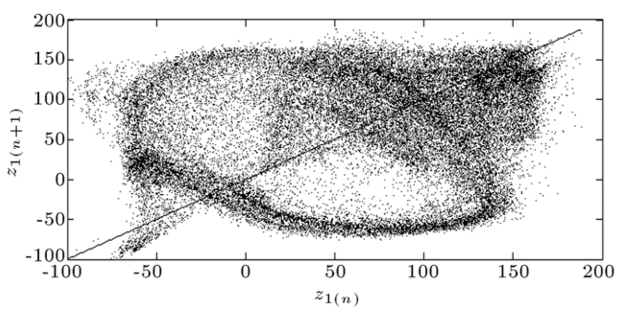


Figure 6: First return map of the Lorenz system with complex variables for $c = 30$, when the Poincaré points are the local maxima of the real part of variable $z$.

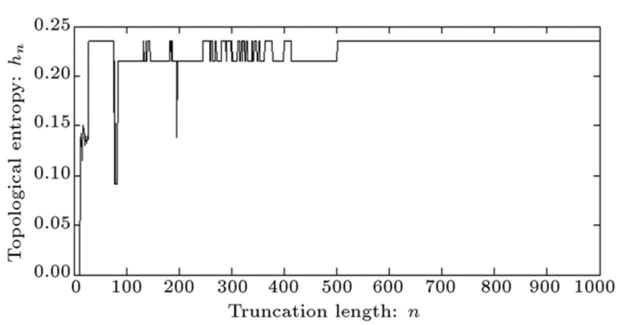


Figure 7: Topological entropy of the first return map to the described Poincaré section of the Lorenz system involving real variables for $c = 30$.

The topological entropy of the first return map for System (2) is greater than the topological entropy of System (1). This shows that the Lorenz system expressed in terms of complex variables is more complex for this value of parameter $c$.

As mentioned in Table 3, the behaviors of System (2) changes for different values of parameter $c$. Figures 9 and 10 demonstrate some phase portraits of System (2) for some of these values when the behaviors are periodic, quasi-periodic, chaotic or hyper-chaotic.

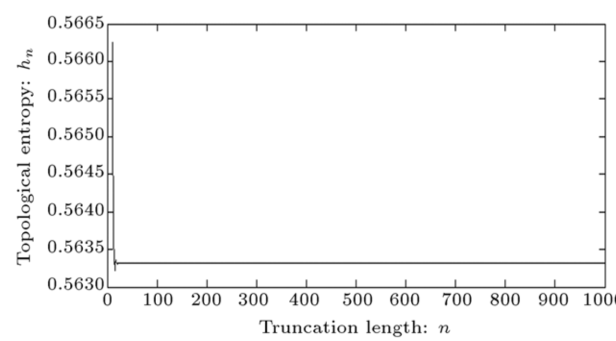


Figure 8: Topological entropy of the first return map to the described Poincaré section of the Lorenz system with complex variables for $c = 30$.

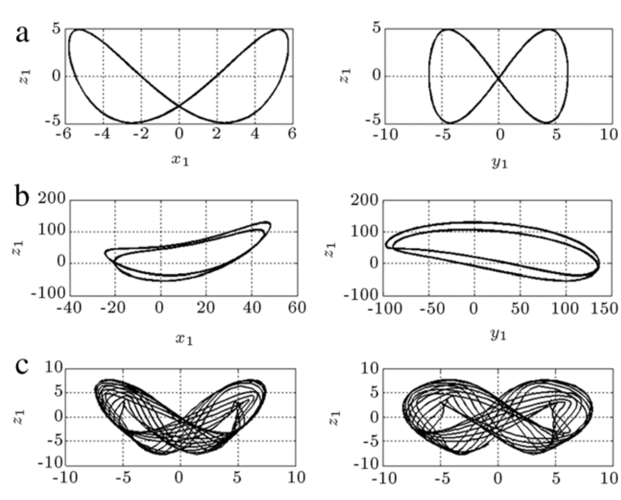


Figure 9: Periodic and quasi-periodic phase portraits of the Lorenz system involving complex variables when $a = 10$, $b = \frac{8}{3}$ and: (a) $c = 3$, (b) $c = 27.7$, and (c) $c = 3.5$.

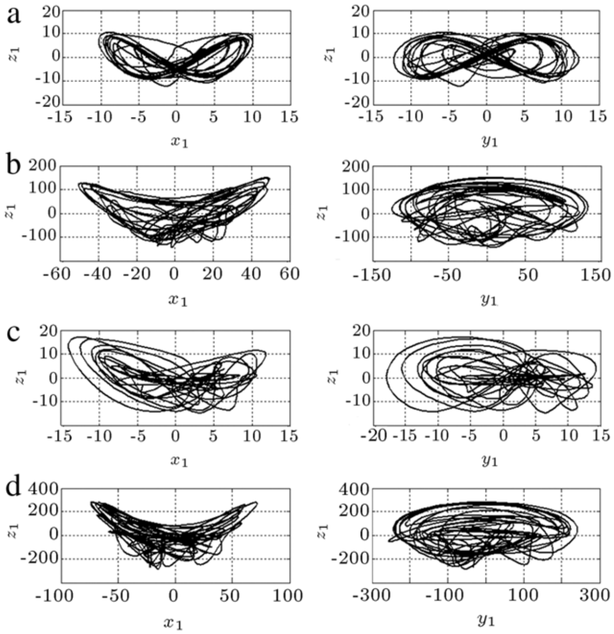


Figure 10: Chaotic and hyper-chaotic behavior of the Lorenz system for $a = 10$, $b = \frac{8}{3}$ *and* (a) $c = 3.7$, (b) $c = 27.8$, (c) $c = 5.1$, and (d) $c = 51$.

## 4. Conclusions and discussion

This paper explains the dynamics of the Lorenz system involving complex variables, and compares it to the Lorenz system with real variables. Both systems have the same structure and features; they are autonomous, non-linear, dissipative with bounded solutions, symmetric with respect to the $z$ axis, and they involve only first order time derivatives. The phase portraits, Lyapunov Exponents spectrums, bifurcation diagrams, first return maps to the Poincare sections and the topological entropies of the systems prove that it is possible to have hyper-chaos in the dual form of the system expressed in terms of complex variables. Indeed, the steady state trajectory of this system can be attracted to a limit cycle, a torus, a chaotic attractor or even a hyper-chaotic attractor as shown in Figures 9 and 10.

Two systems are different in their behavioral and dynamical features; the main differences between the behavioral features of the Lorenz system with complex variables and the Lorenz system with real variables are described as:

1. The existence of a limit cycle in the Lorenz system with complex variables, when the Lorenz system with real variables is attracted to a fixed point or a chaotic attractor.
2. The existence of a torus attractor in the Lorenz system with complex variables, when the Lorenz system with real variables is attracted to a fixed point.
3. A chaotic attractor in the Lorenz system with complex variables, that is more complex than the chaotic attractor in the Lorenz system with real variables. This complex chaotic attractor of the Lorenz system with complex variables exists, even with parameter values in which the Lorenz system with real variables is attracted to a fixed point. Although the phase portrait of the Lorenz system with complex variables in chaotic mode is somehow complex, having just one positive Lyapunov Exponent in these cases proves that the system is in chaotic mode.
4. A complex hyper-chaotic behavior in the Lorenz system with complex variables in certain parameter values, while the Lorenz system with real variables is attracted to a fixed point or have chaotic behavior in the same parameter values. In this situation, the Lorenz system with complex variables has two positive Lyapunov Exponents and the phase portrait of the system is more complex than chaos.

The main dynamical differences between the Lorenz system with complex variables and the Lorenz system with real variables are described as:

1. For some ranges of parameter $c$, the Lorenz system with complex variables has no positive Lyapunov Exponent. For some ranges it has one, and for a wide range of parameter $c$, it has two positive Lyapunov Exponents.
2. Comparing the bifurcations diagrams, we see that the Lorenz system with complex variables has more fixed points in a larger space than the Lorenz system with real variables for the same parameter values.
3. The first return map to the Poincaré section at the local maxima of variable $z$ is unimodal in the Lorenz system involving real variables, while the return map to the equivalent Poincaré section at the local maxima of the real part of variable $z$ for the Lorenz system with complex variables is very complicated and not unimodal.
4. For $c = 30$, the topological entropy of the Poincaré first return map of the Lorenz system with complex variables converges to 0.5633, while it converges to 0.235 for the Lorenz system with real variables. This implies the existence of more complex dynamics in the Lorenz system expressed in terms of complex variables.

The idea that a system involving complex variables has more complicated dynamical behaviors was studied in this article, through simulations and numerical analysis in the Lorenz system. In future, this issue should be expanded to other systems. Also, in order to be able to generalize these findings, a mathematical study should be implemented.

## References

[1] Lorenz, E.N. "Deterministic non-periodic flow", *J. Atmospheric Sci.*, 20, pp. 130–141 (1963).
[2] Dafilisa, P.M., Frascolia, F., Caduschb, J.P. and Lileya, T.J.D. "Chaos and generalized multistability in a mesoscopic model of the electroencephalogram", *Physica D*, 238(13), pp. 1056–1060 (2009).
[3] Wong, K.W., Kwok, B.S.H. and Yuen, C.H. "An efficient diffusion approach for chaos-based image encryption", *Chaos Solitons Fractals*, 41(5), pp. 2652–2663 (2009).
[4] Letellier, C., Aguirre, L.A., Maquet, J. and Gilmore, R. "Evidence for low dimensional chaos in sunspot cycles", *Astronom. Astrophys.*, 449(1), pp. 379–387 (2006).
[5] Wu, J.G., Xia, G.Q., Tang, X., Lin, X.D., Deng, T., Fan, L. and Wu, Z.M. "Time delay signature concealment of optical feedback induced chaos in an external cavity semiconductor laser", *Opt. Express*, 18(7), pp. 6661–6666 (2010).
[6] Wang, X. and Wang, M.A. "Hyperchaos generated from Lorenz system", *Physica A*, 387(14), pp. 3751–3758 (2008).
[7] Jones, C.A., Weiss, N.O. and Cattaneo, F. "Nonlinear dynamos: a complex generalization of the Lorenz equations", *Physica D*, 14(2), pp. 161–176 (1985).
[8] Jia, Q. "Hyperchaos generated from the Lorenz chaotic system and its control", *Phys. Lett. A*, 366(3), pp. 217–222 (2007).
[9] Xu, J., Cai, G. and Zheng, S. "Adaptive synchronization for an uncertain new hyperchaotic Lorenz system", *Int. J. Nonlinear Sci.*, 8(1), pp. 117–123 (2009).
[10] Rossler, O.E. "An equation for hyperchaos", *Phys. Lett. A*, 71(2–3), pp. 155–157 (1979).
[11] Gangadhar, C.H. and Deergha, R.K. "Hyper-chaos based image encryption", *Internat. J. Bifur. Chaos*, 19(11), pp. 3833–3839 (2009).
[12] Gaoa, T. and Chen, Z. "A new image encryption algorithm based on hyper-chaos", *Phys. Lett. A*, 372(4), pp. 394–400 (2008).
[13] Rhouma, R. and Belghitha, S. "Cryptanalysis of a new image encryption algorithm based on hyper-chaos", *Phys. Lett. A*, 372(38), pp. 5973–5978 (2008).

[14] Chen, A., Lu, J., Lu, J. and Yu, S. "Generating hyperchaotic Lü attractor via state feedback control", *Physica A*, 364, pp. 103–110 (2006).
[15] Chen, Z., Yang, Y., Qi, G. and Yuan, Zh. "A novel hyperchaos system only with one equilibrium", *Phys. Lett. A*, 360(6), pp. 696–701 (2007).
[16] Yan, Z. "Controlling hyperchaos in the new hyperchaotic Chen system", *Appl. Math. Comput.*, 168(2), pp. 1239–1250 (2005).
[17] Yassen, M.T. "On hyperchaos synchronization of a hyperchaotic Lü system", *Nonlinear Anal. TMA*, 68(11), pp. 3592–3600 (2008).
[18] Zhou, Q., Chen, Z. and Yuan, Zh. "Blowout bifurcation and chaos-hyperchaos transition in five dimensional continuous autonomous systems", *Chaos Solitons Fractals*, 40(2), pp. 1012–1020 (2009).
[19] Stouboulos, I.N., Miliou, A.N., Valaristos, A.P., Kyprianidis, I.M. and Anagnostopoulos, A.N. "Crisis induced intermittency in a fourth-order autonomous electric circuit", *Chaos Solitons Fractals*, 33(4), pp. 1256–1262 (2007).
[20] Tigan, G. and Opris, D. "Analysis of a 3D chaotic system", *Chaos Solitons Fractals*, 36(5), pp. 1315–1319 (2008).
[21] Wu, W., Chen, Z. and Yuan, Z. "The evolution of a novel four-dimensional autonomous system: among 3-torus, limit cycle, 2-torus, chaos and hyperchaos", *Chaos Solitons Fractals*, 39(5), pp. 2340–2356 (2009).
[22] Clarke, C.K. "Computation of the fractal dimension of topographic surfaces using the triangular prism surface area method", *Comput. Geosci.*, 12(5), pp. 713–722 (1986).
[23] Liebovitch, S.L. and Toth, T. "A fast algorithm to determine fractal dimensions by box counting", *Phys. Lett. A*, 141(8–9), pp. 386–390 (1989).
[24] Hilborn, C.R., *Chaos and Nonlinear Dynamics: An Introduction for Scientists and Engineers*, 2nd ed., Oxford University Press, UK (2000).
[25] Wolf, A., Swift, J.B., Swinney, H.L. and Vastano, J.A. "Determining Lyapunov exponents from a time series", *Physica D*, 16(3), pp. 285–317 (1985).
[26] Szemplinska-Stupnicka, W. "Chaos bifurcations and fractals around us: a brief introduction", *World Scientific Series on Nonlinear Science, Series A*, 47, USA (2003).
[27] Ottino, J.M. "The kinematics of mixing: stretching, chaos, and transport", In *Cambridge Texts in Applied Mathematics*, Cambridge University Press, Cambridge, UK (1989).
[28] Tufaile, A. and Tufaile, A.P.B. "Stretching and folding mechanism in foams", *Phys. Lett. A*, 372(42), pp. 6381–6385 (2008).
[29] Letellier, C., Aguirre, L.A., Maquet, J. and Lefebvre, B. "Analogy between a 10D model for nonlinear wave–wave interaction in a plasma and the 3D Lorenz dynamics", *Physica D*, 179(1–2), pp. 33–52 (2003).
[30] Ubeyli, E.D. and Guler, I. "Statistics over Lyapunov exponents for feature extraction: electroencephalographic changes detection case", *World Acad. Sci. Eng. Tech.*, 2, pp. 132–135 (2005).
[31] Shiriaev, A.S., Freidovich, L.B. and Manchester, I.R. "Can we make a robot ballerina perform a pirouette? orbital stabilization of periodic motions of underactuated mechanical systems", *Annu. Rev. Control*, 32(2), pp. 200–211 (2008).
[32] Duarte, J., Janurio, C. and Martins, N. "Topological entropy and the controlled effect of glucose in the electrical activity of pancreatic $\beta$-cells", *Physica D*, 238(21), pp. 2129–2137 (2009).
[33] Sella, L. and Collins, P. "Computation of symbolic dynamics for one-dimensional maps", *J. Comput. Appl. Math.*, 234, pp. 418–436 (2010).

**Motahareh Moghtadaei** is a Ph.D. degree candidate at Amirkabir University of Technology, Tehran, Iran, from where she received her B.S. and M.S. degrees in Biomedical Engineering. She has authored and/or co-authored several articles in Persian scientific journals in the field of Biomedical Engineering, as well as presented articles in related fields of endeavor at international conferences.

**Mohammad Reza Hashemi Golpayegani** is Professor at the Biomedical Engineering Department of Amirkabir University of Technology, Tehran, Iran, and was Minister of Culture and Higher Education in Iran from 1983 to 1987. He has authored and co-authored numerous papers in Electrical and Biomedical Engineering, and has also been active in the field of the Philosophy of Science and Engineering Education. He is credited with the publication of a number of articles in international scientific journals in the fields mentioned.