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**STUDENT VERSION**

# WATER HAMMER

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## STATEMENT

**Introduction.** In the majority of engineering applications water is treated as an incompressible fluid, in spite of the fact that water is about 100 times more compressible than steel, reinforced concrete, cast iron, and other commonly used structural materials. Water must be confined within rigid walls and subject to very high pressures in order for its compressibility to become important. Two major examples where water compressibility is significant are groundwater flow in confined aquifers and the water hammer. In the following we will focus our attention only on the water hammer phenomenon.

**Physical Description.** A water hammer is caused by the sudden disruption of flow in a pressurized conduit, due to an abrupt closure of a valve, pump shutoff, etc. A water hammer is a violent phenomenon, resulting in the development of very high pressure “heads” (pressure expressed in terms of the height of a water column acting on a unit surface) under which, water becomes compressible and pipe walls deform and, potentially, get damaged. This, creates an elastic wave that bounces back and forth along the pipe in an oscillatory fashion, until the wave is completely diminished by frictional effects. The severity of the water hammer effects can be explained by using the Newton’s second law of motion:

$$F = ma = m\frac{\Delta u}{\Delta t} = m\frac{u_1 - u_2}{\Delta t} = \frac{mu_o}{\Delta t} \qquad (1$$

where F = force, m = mass of flowing water, a = acceleration, $u_1 = u_o$ = initial water velocity, $u_2 = 0$ = terminal water velocity (after the valve closure), and Δt = time duration of valve closure. Therefore, an instantaneous closure time $\Delta t \rightarrow 0$, results in an infinite force $F \rightarrow \infty$.

**Governing Equations.** The governing equations for quantification of the water hammer phenomenon can be established by considering an infinitesimal segment of a pipe as "control volume", CV, (Eulerian description of flow) and applying the principles of mass and momentum balance.

*Equation of Mass Balance.* By utilizing the Taylor series expansion, the mass balance relation for an elementary control volume, Adx,

Flow in – Flow out = Change of fluid mass within the CV

can be written as

$$[\rho u A] - \left[\rho u A + \frac{\partial(\rho u A)}{\partial x}dx\right] = \frac{\partial(\rho A)dx}{\partial t} \qquad (2)$$

where ρ = density of the water considered as compressible, u = average cross-sectional velocity, A = interior cross section area of the pipe and dx = a constant infinitesimal pipe length. The first term in square brackets is the rate of fluid mass entering the CV, the second term in brackets is the rate of fluid mass exiting the CV (obtained using the first-order term of the Taylor series expansion), while the term on the other side is the change of mass in time within the CV.

After some cancellations, expansion of differentials using the product rule of differentiation, rearrangement of the terms, and division by ρA, the student should be able to derive (3) (Hint: dx is constant since it is the length of a fixed size CV.)

$$\frac{1}{\rho}\frac{\partial \rho}{\partial t} + \frac{u}{\rho}\frac{\partial \rho}{\partial x} + \frac{1}{A}\frac{\partial A}{\partial t} + \frac{u}{A}\frac{\partial A}{\partial x} + \frac{\partial u}{\partial x} = 0. \qquad (3)$$

In (3), the first and third terms indicate the local time changes of the variables, while the second and fourth terms indicate the advection changes of the variables at an adjacent point. Thus, written in terms of total derivatives D/Dt (including both local and advection effects) yields

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \frac{1}{A}\frac{DA}{Dt} + \frac{\partial u}{\partial x} = 0. \qquad (4)$$

The first term in (4) quantifies the water compressibility effects, while the second term quantify the elastic deformation of the pipe. Thus, these two terms can be expressed as follows

$$\frac{1}{\rho}\frac{D\rho}{Dt} = \frac{1}{E_f}\frac{Dp}{Dt} = \frac{1}{E_f}\left(\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x}\right), \qquad (5)$$

$$\frac{1}{A}\frac{DA}{Dt} = \frac{D_o k(\varepsilon)}{eE_p}\frac{Dp}{Dt} = \frac{D_o k(\varepsilon)}{eE_p}\left(\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x}\right), \qquad (6)$$

where p = water pressure, $D_o$ = interior pipe diameter, e = pipe wall thickness, $E_f$ = bulk modulus of elasticity of the fluid, $E_p$ = Young's modulus of elasticity of the pipe material, ε = Poisson's ratio of the wall pipe material which is included in the coefficient k = k(ε) [Houghtalen *et al*., 2017]. The moduli of elasticity and the Poisson ratio are experimentally defined quantities. In case the longitudinal deformation of the pipe, due to expansion or contraction of the pipe's cross section area (Poisson ratio) is negligible, then k(ε) =1. For pipes anchored at both ends k=0.9375, while for pipes with expansion joints k=0.825. By substituting (5) and (6) in (4), the student should show that it results in (7)

$$\frac{1}{\rho}\frac{Dp}{Dt} + c^2\frac{\partial u}{\partial x} = 0, \qquad (7)$$

where c = speed of the pressure (elastic) wave also called celerity c, given as

$$c = \sqrt{\frac{1}{\rho}\left(\frac{E_f}{1 + \frac{E_f}{E_p}\frac{D_o k(\varepsilon)}{e}}\right)} \qquad (8)$$

It should be noted, that the speed of the elastic wave may exceed multiple times the speed of sound, with values being in the range of more than 1,000 m/s. The elastic wave travels back and forth through the pipe, with reducing pressure effects in time, due to frictional damping. Considering that the pressure head is h = p/γ (γ = gρ = specific weight), from (7) the student should derive (9) (Hint: expand the total derivative, substitute p with gρh and treat the specific weight as constant.)

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial \mathrm{x}} + \frac{c^2}{g}\frac{\partial u}{\partial x} = 0. \qquad (9)$$

In the S.I. system of units, pressure is in N/m$^2$, water density is 998 kg/m$^3$, acceleration due to gravity is 9.8 m/s$^2$ and the pressure head is in meters.

Equation (9) is the most general equation that describes the continuity equation for the case of one-dimensional, unsteady-state flow in a pressurized conduit.

*Equation of Momentum Balance.* By utilizing the Taylor series expansion, the momentum balance relation (Newton's 2nd Law) for an elementary control volume Adx:

Summation of forces = Change of momentum within the CV

can be written as

$$pA + p\frac{\partial A}{\partial x}dx - \left(pA + \frac{\partial (pA)}{\partial \mathrm{x}}dx\right) - \tau_o \pi D_o dx = \rho A dx\frac{Du}{Dt} \qquad (10)$$

where $\tau_o$ = frictional shear stress. The three first terms in (10) are pressure forces acting at the entrance and exit of the control volume, the fourth is the frictional force acting throughout the internal wall surface, and the last one is the change of momentum. After some cancellations, term expansions and rearrangements, the student should show that (10) results in

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial \mathrm{x}} + \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{4\tau_o}{\rho D} = 0. \qquad (11)$$

Furthermore, considering that for turbulent flow, experimentally it was found that

$$\frac{\tau_o}{\rho} = \frac{f}{8}u|u| \qquad (12)$$

where f = Darcy-Weisbach friction coefficient, (11) in terms of pressure head h, reads

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial \mathrm{x}} + g\frac{\partial h}{\partial x} + \frac{fu|u|}{2D} = 0. \qquad (13)$$

Equation (13) is the most general equation that describes the momentum equation for the case of one-dimensional, unsteady-state flow in a pressurized conduit.

*System of Governing Equations.* The system of Equations (9) and (13) is a pair of nonlinear PDEs of the hyperbolic type that fully describe the one dimensional evolution of the water hammer phenomenon. The system is an initial-boundary problem which for completeness requires assignment of appropriate initial and boundary conditions.

*Initial and Boundary Conditions.* The initial conditions are the steady-state velocity and the linear pressure distribution from the reservoir surface elevation to zero (gage pressure) at the outlet tip of the pipe. Using the energy equation balance between the water surface of the reservoir and the pipe exit

$$H_o + z_{en} = \frac{u^2}{2g} + \frac{p}{\gamma} + z_{ex} + h_f \qquad (14)$$

where $H_o$ = constant head of the reservoir measured vertically from the pipe entrance, $z_{en}$ , $z_{ex}$ = vertical distance of the pipe entrance and pipe exit from some arbitrary reference datum (Figure 1), and $h_f$ = total frictional head loss, estimated by the Darcy-Weisbach equation as

$$h_f = f\frac{L}{D_o}\frac{u^2}{2g} \qquad (15)$$

where L = pipe length.

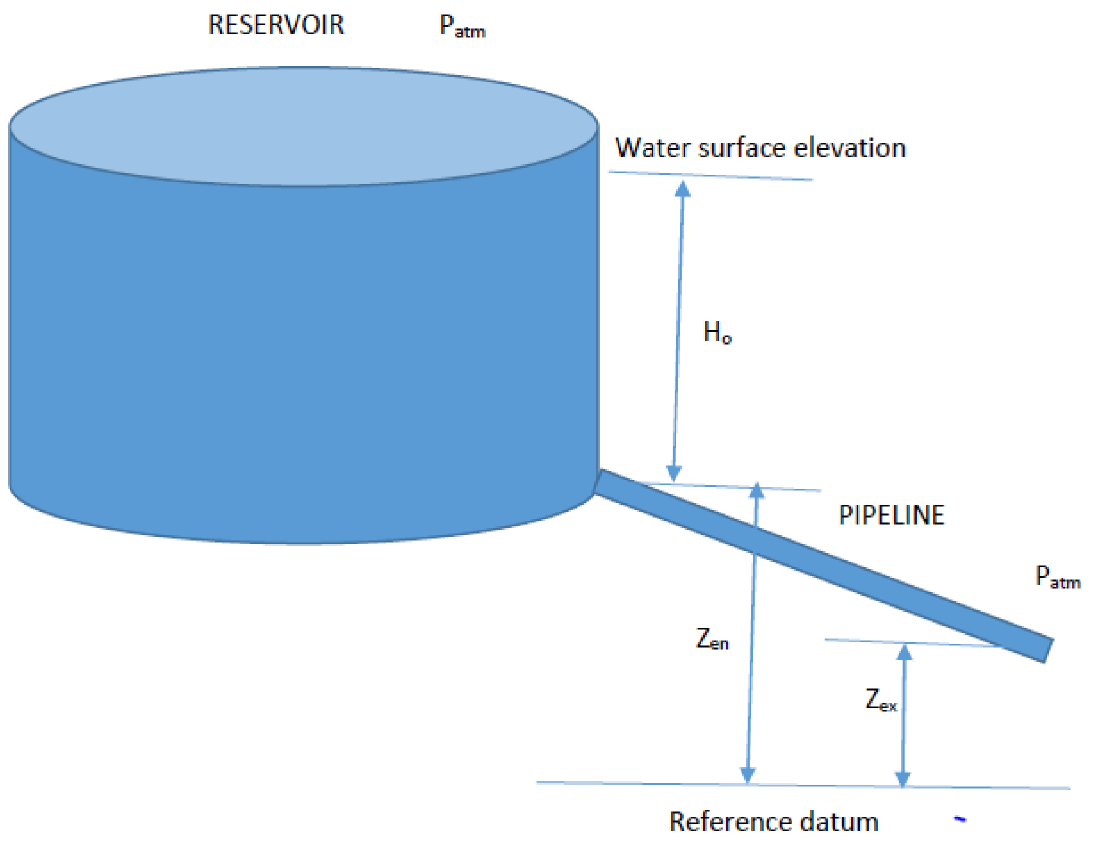


Figure 1. Schematic representation of the reservoir pipe system.

The atmospheric pressure appears at both sides of (14), and it is cancelled. Thus, initially at the exit point $p = p_{atm} = 0$, and the velocity can be found by combining (14) and (15) as

$$u = \sqrt{\frac{2g(H_o + z_{en} - z_{ex})D_o}{D_o + fL}}. \qquad (16)$$

For the case of a horizontal pipe $z_{en} - z_{ex} = 0$.

The upstream boundary condition is defined by the reservoir constant head $H_o$, while the downstream condition is given by the reducing flow velocity according to the schedule of valve closure. After full closure of the valve the velocity at the end is set equal to zero.

*Numerical Solution*. Due to the complexity of the governing equations (9) and (13), the system can only be solved numerically for the variables u(x,t) and h(x,t). For that purpose, the staggered Arakawa C-grid has been employed [Arakawa and Lamb, 1977]. This is an explicit finite differences scheme, where the velocity is estimated first from the momentum equation and then the pressure head is estimated using the continuity equation. It should be noted however, that the velocities are estimated at the sides of each discretized segment while the pressure head at the center of the segment; thus the solution proceeds in a staggered fashion. The Arakawa C-grid scheme ensures that mass is conserved during the numerical solution.

Without losing too much of the solution accuracy and for simplicity, the system can be linearized by dropping the advection effects (2nd term in both (9) and (13)). The remaining nonlinear frictional term in the momentum equation is “treated” numerically, by using the previous time-step already calculated values of the velocity. In this way, the nonlinear term is a known quantity. Thus, the finite difference equations are

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + g\frac{h_i^n - h_{i-1}^n}{\Delta x} + \frac{f|u_i^n|u_i^n}{2D_o} = 0, \qquad (17)$$

$$\frac{h_i^{n+1} - h_i^n}{\Delta t} + \frac{c^2}{g}\frac{u_{i+1}^{n+1} - u_i^{n+1}}{\Delta x} = 0. \qquad (18)$$

The subscript, i, refers to the location of the finite segment along the pipe, while the superscript, n, to time. However, for the pressure head the index i refers to the discretized segment sequence number, while for the velocity i is indicative of the left-hand side boundary of the same segment and i+1 of the right-hand side, when flow direction is from left to right (Figure 2).

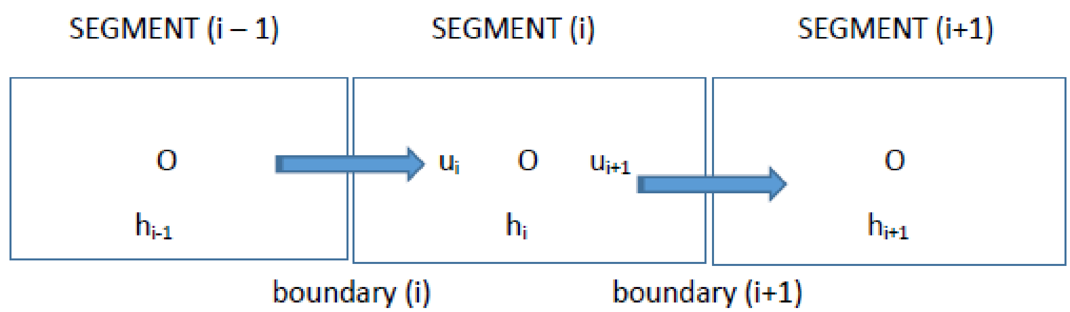


Figure 2. Computational location of u(x,t) and h(x,t) [o is the center of the finite segment].

The final discretized linearized solution scheme reads:

$$u_i^{n+1} = u_i^n - g\frac{\Delta t}{\Delta x}(h_i^n - h_{i-1}^n) - \frac{f\Delta t}{2D_o}|u_i^n|u_i^n \quad 19)$$

$$h_i^{n+1} = h_i^n - \frac{c^2}{g}\frac{\Delta t}{\Delta x}\left(u_{i+1}^{n+1} - u_i^{n+1}\right) \qquad (20)$$

Although the Courant-Friedrichs-Levi (CFL) stability criterion does not apply rigorously, the criterion should be used for selection of the size of spatial and temporal discretization steps in order to avoid numerical instability.

$$c \leq \frac{\Delta x}{\Delta t} \qquad (21)$$

If the advection terms are included, then the full (9) and (13) are used and the finite differences scheme reads

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + u_i^n\frac{u_{i+1}^n - u_i^n}{\Delta x} + g\frac{h_i^n - h_{i-1}^n}{\Delta x} + \frac{f|u_i^n|u_i^n}{2D_o} = 0, \qquad (22)$$

$$\frac{h_i^{n+1} - h_i^n}{\Delta t} + u_i^{n+1}\frac{h_i^n - h_{i-1}^n}{\Delta x} + \frac{c^2}{g}\frac{u_{i+1}^{n+1} - u_i^{n+1}}{\Delta x} = 0, \qquad (23)$$

resulting in the explicit solution

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}u_i^n(u_{i+1}^n - u_i^n) - g\frac{\Delta t}{\Delta x}(h_i^n - h_{i-1}^n) - \frac{f\Delta t}{2D_o}|u_i^n|u_i^n, \qquad (24)$$

$$h_i^{n+1} = h_i^n - \frac{\Delta t}{\Delta x}u_i^{n+1}\,(h_i^n - h_{i-1}^n) - \frac{c^2}{g}\frac{\Delta t}{\Delta x}\left(u_{i+1}^{n+1} - u_i^{n+1}\right). \qquad (25)$$

By solving (19) - (20) and (24) - (25) the insignificance of the advection effects becomes evident.

*Application.* The linearized model (19) – (20), was applied to a L = 4.8-km long horizontal circular steel pipe ($E_p = 1.9 \times 10^{11}$ N/m$^2$) having an interior diameter of $D_o$ = 1 m and wall thickness of e = 1.25 cm. The pipe is carrying water ($\rho$ = 998 kg/m$^3$, $E_f = 2.2 \times 10^9$ N/m$^2$) from a reservoir with constant head $H_o$ = 15 m (measured vertically from the pipe entrance) and discharges freely in the open air. The longitudinal strain is neglected ($k(\varepsilon)$ =1) and the friction coefficient is taken as f = 0.05. Based on the given data, the steady-state velocity is estimated as u = 1.12 m/s (16) and the celerty as c = 1,070 m/s (8).

Spatial and temporal steps are taken as $\Delta x$ = 60 m and $\Delta t$ = 0.02 s. The selected steps are compliant with the Courant-Friedrichs-Levi criterion (21): c = 1,070 m/s < 3,000 m/s (60 m/0.02 s). The simulation results included three different closure times, of the downstream end valve, of 5, 15 and 35 seconds. During closure time the velocity was linearly reduced from 1.12 m/s to zero. The computations were carried for a simulation period of three minutes (180 s).

Figure 3 shows the time record of oscillatory pressure heads (expressed in meters) at the end of the pipe, while Figure 4 shows the maximum recorded pressure heads throughout the length (80 steps x 60 m/step = 4,800 m) of the pipe during the phenomenon (Dx = $\Delta x$ = 60 m).

The impact of the valve closure time on the pressure development and the damping due the frictional effects are evident. The numerical computations and the plots were conducted by using the MATLAB (2021a) [Hunt *et al*., 2016].

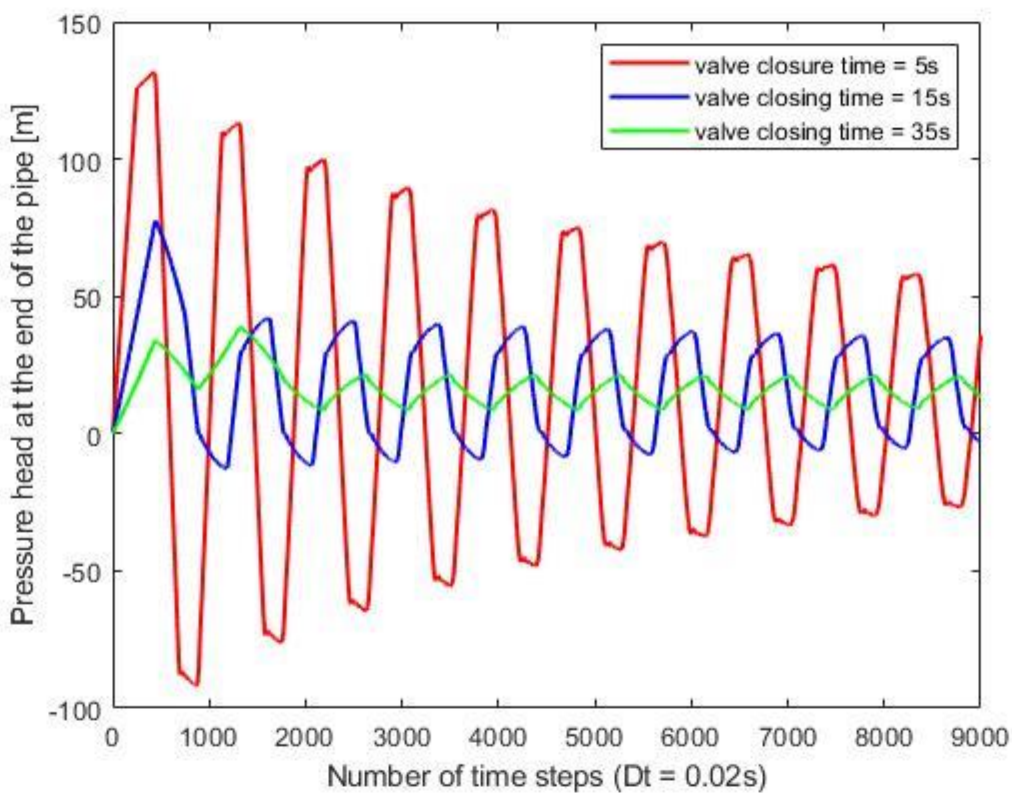


Figure 3. Pressure head oscillations at the downstream end of the pipe.

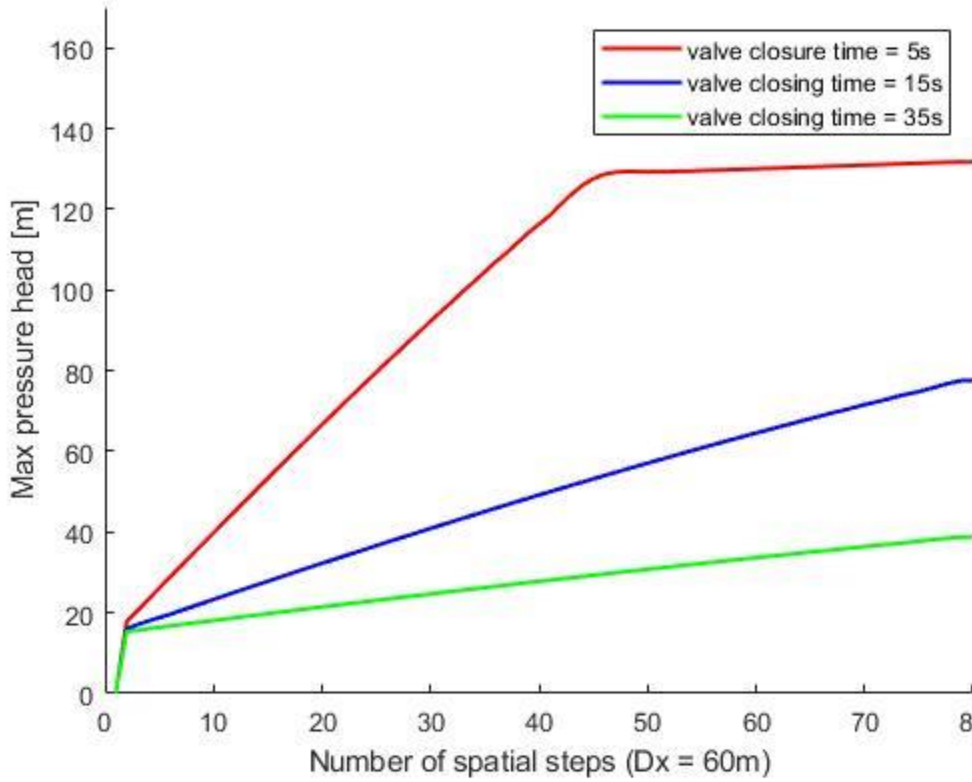


Figure 4. Maximum pressure heads recorded over the length of the pipe.

## ACTIVITIES

**Activity 1**: Make yourself familiar with the problem under consideration. Identify all of the constants and parameters involved in the analysis of the water hammer phenomenon. Make a list of them and provide their numerical value or the practical range of their expected values, e.g., $E_f = 2.2 \times 10^9$ N/m$^2$, $0.008 < f < 0.1$, etc. This list could be used for sensitivity analysis and design purposes after the model is built. Think of different fluids, pipe materials and geometric characteristics. Discuss your findings in a one-page, single-spaced report. [10 points]

(Note: the points assigned are to signify relevant weights of the Activities, not absolute values for any individual grading scheme)

**Activity 2**: Take a critical look at the governing PDEs and identify the physical meaning of each term. What is the difference between Eulerian and Lagrangian description? Which are the terms that may be of minor importance to the development of the phenomenon? What is the significance of the PDEs classification as hyperbolic, as compared to elliptic or parabolic

PDEs? From the governing equations (9) and (13) make the appropriate assumptions and derive the simple wave equations

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad or \qquad \frac{\partial^2 h}{\partial t^2} - c^2 \frac{\partial^2 h}{\partial x^2} = 0. \quad (26)$$

Discuss your findings in a one-page, single-spaced report. [10 points]

**Activity 3**: Take a critical look at the numerical treatment of the governing equations. What are the pros and cons of the Arakawa scheme? What about its consistency, convergence, and stability? If we want to keep the advection terms how can we proceed numerically? Discuss your findings in a one-page, single-spaced report. [10 points]

**Activity 4**: Based on the Arakawa scheme, develop a computer code. Make assumptions as needed; change parameters and monitor the effect of the changes in the water hammer behavior. Also, test the model's reliability by changing the spatial or temporal steps. More specifically, using the data and results shown in the *Application* (see page 9) as a reference point what changes do you observe if:

- Pipe material is lead
- Pipe diameter is 40 cm
- Pipe thickness is 1 cm
- Pipe frictional coefficient is 0.01
- Pipe length is 2.4 km
- Pipe has a slope of 2:1000 (vertical:horizontal)
- Transported fluid is crude oil
- Increase the time step by increments of 0.02 s e.g., 0.04, 0.06, 0.08, etc.; see when the scheme becomes unstable
- Include the advection terms and see if there are any noticeable changes in the pressure heads.

Combine briefly Activities 1 to 3, and data from the computer simulations in Activity 4, into a comprehensive written report. [50 points].

**Activity 5.** Based on your overall findings, make a 12-slide presentation, and prepare a 10-minute oral presentation. Make yourself sound like an expert in the water hammer phenomenon. [20 points]

## REFERENCES


Arakawa A. and Lamb V.R. 1977. *Computational design of the basic dynamical processes of the UCLA general circulation model.* Methods in Computational Physics: Advances in Research and Applications, 17: 173-265.

Houghtalen Robert J., Akan A. Osman, and Hwang Ned A.J. 2017. *Fundamentals of Hydraulic Engineering Systems*. Boston MA: Pearson, 5th edition.

Hunt Brian R., Lipsman Ronald L., and Rosenberg Jonathan M. 2016. *A Guide to MATLAB for Beginners and Experienced Users*. Cambridge: Cambridge University Press, 3rd edition.