www.simiode.org
SIMIODE A SYSTEMIC INITIATIVE FOR MODELING INVESTIGATIONS & OPPORTUNITIES WITH DIFFERENTIAL EQUATIONS

STUDENT VERSION
Modeling the Velocity of a Pull-Back Toy

Brody Dylan Johnson
Saint Louis University
Saint Louis, Missouri USA

STATEMENT

A pull-back toy (as shown in Figure 1) is typically a small car or truck powered by a clockwork motor [1]. A flat spiral spring in the pull-back mechanism is wound as the toy vehicle is pushed backwards and this spring provides a locomotive force when the toy is released [2]. In this project, a mathematical model will be developed to predict the motion of a pull-back toy using real data and, ultimately, to estimate the maximum speed and travel distance of the toy.

The pull-back motor has an ingenious design that allows the internal spring to be wound over a relatively short distance as the toy is moved in reverse, yet it provides locomotive force over a substantially longer distance as the toy rolls forward. One possibly unexpected consequence of this design is that the locomotive force of the pull-back motor stops once the toy has traveled a specific distance (as opposed to a specific amount of time). Once the spring has completely unwound, the toy will begin to coast and gradually slows down as a result of various frictional losses. This unique characteristic of the pull-back motor must be treated carefully in the mathematical model.

Figure 1. A pull-back toy obtained as a cereal box prize, circa 2006.

A variety of physical quantities could enter into play in the analysis of a pull-back toy, e.g., the mass of the vehicle and wheels, the rotational inertia of the wheels, the radius of the wheels, the stiffness of the internal spring, the gear ratios of the pull-back mechanism, etc. It would be a difficult task to determine precise values for all of these quantities. Fortunately, it is possible to build a realistic model describing the motion of the pull-back toy without any knowledge of these physical constants. This fact will be explained in detail in the Section 1 and may go beyond the scope of a typical course in ordinary differential equations. However, this explanation can be omitted without any detriment to the project, allowing students to either skim or skip Section 1 before proceeding to Section 2.

MATERIALS

Sample data obtained from experiments with a real pull-back toy will be provided. However, students are also encouraged to conduct experiments with their own pull-back toy. To do so, students are advised to have the following materials on hand:

- A pull-back toy;
- A digital video camera;
- A tape measure;
- A roll of masking tape.

1 The Mathematical Model

Newton's Second Law is the key tool in the development of a mathematical model for the pull-back toy. A simple schematic drawing of a typical pull-back toy is provided in Figure 2. The effect of the pull-back motor is to apply a moment $M(x,t)$ to the rear axle of the toy, which is counteracted by a frictional force $f(x,t)$ between the ground and the wheels. It will be assumed that the moment M cannot overcome the frictional force f, which means the pull-back toy will be subject to *pure rolling*, i.e., the wheels will not slip on the ground.

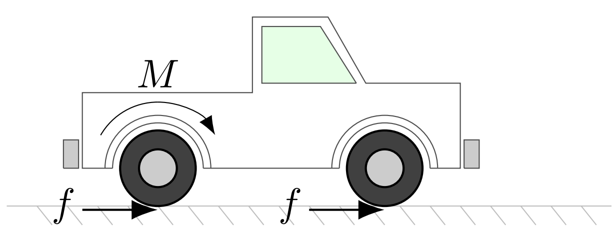

Figure 2. Schematic of Pull-Back Toy with External Forces.

The pure rolling assumption guarantees that distance traveled by the pull-back toy, $x(t)$, will be directly proportional to $\theta(t)$, the angle of rotation of the wheels. In particular,

$$x(t) = r\theta(t), \tag{1}$$

where r is the radius of the wheels. In order to account for the rotation of the wheels and the linear motion of the toy, Newton's Second Law must be applied in both the rotational and linear forms.

Applying the rotational form of Newton's Second Law to the wheels (all four combined) leads to the equation

$$I\frac{d^2\theta}{dt^2} = M(x,t) - rf(x,t), \tag{2}$$

where I represents the combined rotational inertia of the wheels. Note that, in (2), $f(x,t)$ represents the combined frictional force applied to all four wheels. This frictional force also acts to accelerate the pull-back toy horizontally and is transferred to the body of the pull-back toy via the axles. Hence, application of the linear form of Newton's Second Law to the pull-back toy produces the equation

$$m\frac{d^2x}{dt^2} = f(x,t), \tag{3}$$

where m represents the mass of the pull-back toy (including the wheels). Together, (2) and (3) describe the motion of the pull-back toy.

The equations of motion (2) and (3) are relatively simple, but, unfortunately, it is not practical to quantify all of the physical constants involved. Luckily, the pure rolling assumption permits a critical simplification because the relationship between x and θ can be used to eliminate the frictional force f. Indeed, multiplying (3) by r, adding the result to (2), and applying (1) leads to a single equation of motion,

$$\left(\frac{I}{r} + mr\right)\frac{d^2x}{dt^2} = M(x,t). \tag{4}$$

It is evident from the equation of motion that the acceleration of the pull-back toy is proportional to the moment applied by the internal spring. There is one further adjustment that should be made to the equation of motion in order to make it as realistic as possible. In order to account for air resistance and other frictional losses, it will be assumed that the speed of the pull-back toy decays exponentially in the absence of the moment from the pull-back motor. These observations lead to an equation of motion with the following form,

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} = \widetilde{M}(x,t),$$

where k is the decay rate for the velocity of the pull-back toy and

$$\widetilde{M}(x,t) := \frac{M(x,t)}{\frac{I}{r} + mr} = \frac{M(x,t)r}{I + mr^2},$$

represents the applied moment of the pull-back motor normalized by appropriate physical constants.

Notice that the normalized moment due to the pull-back motor has been written in the form $\widetilde{M}(x,t)$, meaning that it could depend explicitly on either or both of the variables x and t. This moment is produced by the unwinding of the internal spring within the pull-back motor. The spring unwinds as the wheels rotate and the car rolls forward, which means the spring tension is a function of x and has no *explicit* dependence on t. Therefore, the normalized moment shall be written hereafter as $\widetilde{M}(x)$. Moreover, the pull-back motor produces no moment once the internal spring has unwound fully, implying that $\widetilde{M}(x) = 0$ for sufficiently large x.

It is not clear how quickly the torque output of the pull-back motor decreases as the internal spring unwinds. Two simple models will be considered. The first approach will be referred to as the *constant-torque* model and assumes that $\widetilde{M}(x)$ has the form

$$\boxed{\widetilde{M}(x) = \begin{cases} M_0 & 0 \leq x \leq x_0 \\ 0 & x > x_0. \end{cases},} \tag{5}$$

where x_0 is the distance at which the internal spring will have completely unwound. The graph of $\widetilde{M}(x)$ for the constant-torque model is shown in Figure 3. The second approach is based on a *linear-torque* model, which assumes that $\widetilde{M}(x)$ takes the form

$$\boxed{\widetilde{M}(x) = \begin{cases} M_0 \left(1 - \frac{x}{x_0}\right) & 0 \leq x \leq x_0 \\ 0 & x > x_0. \end{cases}.} \tag{6}$$

The graph of $\widetilde{M}(x)$ for the linear-torque model is shown in Figure 4.

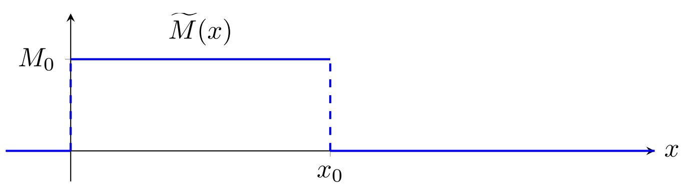

Figure 3. The form of the moment function for the constant-torque model.

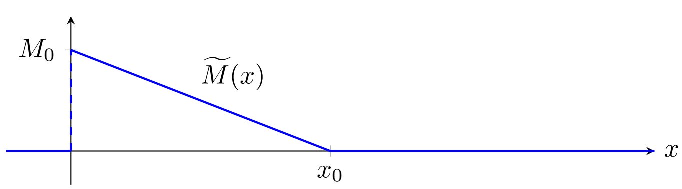

Figure 4. The form of the moment function for the linear-torque model.

Under all of the above assumptions, the position of the pull-back toy will be modeled by the second-order differential equation

$$x'' + kx' = \widetilde{M}(x), \tag{7}$$

where $\widetilde{M}(x)$ obeys either (5) or (6). It is now clear that each model depends on just three physical constants: M_0, x_0, and k. The constant M_0 quantifies the maximum output of the pull-back motor, while the constant x_0 represents the distance over which the internal spring completely unwinds. Finally, the constant k accounts for the decay of the velocity of the pull-back toy due to air resistance and other frictional losses.

2 The Empirical Model

In Section 1, a mathematical model was derived to describe the horizontal position of a pull-back toy after it is fully wound and released. It was shown that the horizontal position of the pull-back toy, $x(t)$, can be modeled by the second-order differential equation $x'' + kx' = \widetilde{M}(x)$, where $\widetilde{M}(x)$ is given by (5) for the constant-torque model or (6) for the linear-torque model. In either case, the model employs three physical constants: the constant x_0 describes the wind-down distance of pull-back motor, M_0 quantifies the moment generated by the pull-back motor, and k represents the decay rate of the toy's speed due to frictional losses. The goal of this section is to conduct experiments that will lead to estimates for each of these constants.

2.1 Estimation of x_0

The second paragraph of the problem statement described one of the key innovations of the pull-back motor, namely, that the motor can be wound up by pushing the toy backwards over a short distance, yet the motor supplies torque to the wheels over a much longer distance when the toy is in motion. The shorter distance over which the toy is pushed backwards will be referred to as the *wind-up distance*, while the longer distance over which the pull-back motor unwinds will be called the *wind-down distance*. The wind-down distance, denoted by x_0, indirectly describes the amount of time that the pull-back motor will supply torque and is, therefore, critical to estimates of the top speed and travel distance for a pull-back toy. Fortunately, it is possible to estimate x_0 without taking the toy apart to examine the internal components of its pull-back motor. For the toy shown in Figure 1, the wind-down distance x_0 is approximately 218`cm`. In contrast, the wind-up distance for the toy is roughly 20`cm`, corresponding to a ratio of about 10.9:1.

Student Task: Use the procedure below to obtain an estimate of x_0. Students who are unable to conduct their own experiment can skip this task and use the given value for x_0.

1. **Wind the pull-back toy up by pushing it backwards.**

   A series of noticeably louder/stronger clicks will be heard/felt when the internal spring is fully wound.

2. **Place a mark at the starting location of the pull-back toy.**

   The wind-down distance x_0 could be anywhere from a couple of feet up to ten or more feet. Make sure that there is enough space for the toy to wind down completely.

3. **Keeping one hand in contact with the front of the toy, allow the toy to make a series of short advances until the motor has completely wound down.**

   This process will take a little while and the key is to regularly stop the car so that it is unable to coast after the torque from pull-back motor is exhausted. It may take several attempts to perform this step smoothly.

4. **Place a mark at the ending location of the pull-back toy.**

   It would be wise to use a different mark for each trial, e.g., a piece of masking tape with the trial number written on it.

5. **Measure the distance between the two marks, which provides an estimate of x_0.**

   Repeat the entire procedure two or three times, making sure the results are consistent.

Questions:

1. How does your estimate of x_0 compare to your expectations?
2. Modify the above procedure to measure the wind-up distance (defined above) for the pull-back toy and denote this distance by x_1. What is the ratio of x_0 to x_1? How does it compare to the ratio given for the toy shown in Figure 1?
3. How consistent were the estimates obtained in the different trials? Did anything happen during your trials that could have affected the estimate, e.g., did the wheels slip at any point?

2.2 Estimation of M_0 and k

It is now time to put the pull-back toy in motion and acquire data that can be used to complete the empirical model. This will be accomplished by video recording the motion of a fully-wound toy as it accelerates from rest. Frame-by-frame analysis of the video will lead to position data for the toy from which the velocity can be estimated. Numerical methods will be used to compare the solution of the mathematical model (7) with the experimental velocity data, leading to estimates for the parameters M_0 and k.

Student Task: Use the following procedure to collect data for the estimation of M_0 and k. Students who are unable to conduct their own experiment can use the data in Table 1.

1. **Use the measuring tape and masking tape to establish a series of regularly-spaced markers on the test surface.**

   Intervals of 5 `cm` are recommended for the first 50 `cm` with intervals of 10 `cm` thereafter. The total length of the test area should be at least 150 `cm`.

2. **Use the video camera to record two or three experiments in which the fully-wound toy accelerates from the starting mark into the test area.**

   Try to keep the camera relatively still during the recording and keep the starting mark and as many of the subsequent marks in the field of view throughout the recording. Only one good experiment is needed, so repeat the experiment until everything goes smoothly and a nice, steady video is obtained.

3. **Create position data from your video using a computer.**

   Advance through the video, frame by frame, and record the number of frames that have elapsed up to the moment when the front of the toy crosses each mark on the test surface. Frame 0 should correspond to the moment when the toy is released from the starting mark. Note the distanced traveled for each mark. Many digital cameras record 30 frames per second, in which case the time for each data point can be calculated as: `time` = `(frame number)`/30. Make sure to verify the frame rate and adjust this formula accordingly, as some cameras use 60 or 120 frames per second. (The video used to produce Table 1 includes 60 frames per second.)

Table 1. Time and Position Data for the Pull-Back Toy Depicted in Figure 1.

Time (s)	0.000	0.250	0.350	0.433	0.517	0.583	0.650
Position (cm)	0	5	10	15	20	25	30
Time (s)	0.700	0.700	0.800	0.850	0.950	1.033	1.117
Position (cm)	35	40	45	50	60	70	80
Time (s)	1.200	1.267	1.333	1.417	1.483	1.550	1.617
Position (cm)	90	100	110	120	130	140	150

At this point, students should have a table of time and position data for a pull-back toy taken while the fully-wound toy is accelerating from rest. The next task involves fitting the second-order mathematical model to this data to obtain estimates for M_0 and k. The improved Euler method will be used to estimate the solution of the mathematical model for any choice of the constants k, x_0, and M_0. In order to use the improved Euler method, the second-order differential equation (7) must be converted into a first-order system. Let $v = x'(t)$ represent the velocity of the pull-back toy, so that $v'(t) = x''(t)$. With this definition, (7) is equivalent to the system

$$\begin{aligned} \frac{dx}{dt} &= v \\ \frac{dv}{dt} &= \widetilde{M}(x) - kv. \end{aligned} \tag{8}$$

The improved Euler method will allow for the estimation of x and v at regularly spaced times $t_n := n\Delta t$, where $\Delta t > 0$ is the step-size and $n \geq 1$ is an integer index for the estimates. Letting x_n and v_n represent the estimates of $x(t_n)$ and $v(t_n)$, the procedure shown in Figure 5 can be used to calculate x_{n+1} and v_{n+1} from x_n and v_n. Notice that because the pull-back toy begins at rest in this experiment, the initial conditions must be $x_0 = 0$ and $v_0 = 0$.

Student Task: Use a spreadsheet to estimate M_0 and k from the experimental data collected earlier.

1. **Set up columns for all of the required variables and enter the observed time and position data.**

   Seven columns are needed: (A) Frame, (B) Time t, (C) Position $x(t)$-observed, (D) Velocity $v(t)$-observed, (E) Position $x(t)$-predicted, (F) Velocity $v(t)$-predicted, (G) Squared Error. Use the sum of the squared error for v in the last column. (See Figure 6 for a screenshot.)

2. **Calculate the observed velocity data from the observed position data.**

   The observed velocity at for the nth data point should be estimated from the position data as follows:
   $$v_n = \frac{\Delta x}{\Delta t} = \frac{x_{n+1} - x_{n-1}}{t_{n+1} - t_{n-1}}.$$

Improved Euler Method for (8):

1. Compute initial slope estimates:

$$\begin{aligned} m_1 &= v_n \\ n_1 &= \widetilde{M}(x_n) - kv_n. \end{aligned}$$

2. Compute a temporary estimate for x and v:

$$\begin{aligned} \tilde{x} &= x_n + m_1 \Delta t \\ \tilde{v} &= v_n + n_1 \Delta t. \end{aligned}$$

3. Compute final slope estimates:

$$\begin{aligned} m_2 &= \tilde{v} \\ n_2 &= \widetilde{M}(\tilde{x}) - k\tilde{v}. \end{aligned}$$

4. Compute x_{n+1} and v_{n+1}:

$$\begin{aligned} x_{n+1} &= x_n + \left(\frac{m_1 + m_2}{2}\right) \Delta t \\ v_{n+1} &= v_n + \left(\frac{n_1 + n_2}{2}\right) \Delta t. \end{aligned}$$

Figure 5. The Improved-Euler Method applied to the system (8).

	A	B	C	D	E	F	G	H
1	Acceleration Data				Model			
2		(seconds)	(cm)	(cm/s)	(cm)	(cm/s)		
3	frame	time	position	velocity	distance	velocity	sq_error	M_o
4	0	0.000	0	0.0	0.00	0.00	0.00	100.0
5	15	0.250	5	28.6	2.99	23.39	26.86	k
6	21	0.350	10	54.5	5.76	31.81	516.94	0.500
7	26	0.433	15	60.0	8.69	38.40	466.39	x_o
8	31	0.517	20	66.7	12.15	44.61	486.50	218.0
9	35	0.583	25	75.0	15.28	49.29	660.88	SSE
10	39	0.650	30	85.7	18.71	53.72	1023.48	37498.1
11	42	0.700	35	100.0	21.48	56.88	1859.42	dt
12	45	0.750	40	100.0	24.40	59.89	1608.56	0.017
13	48	0.800	45	100.0	27.47	62.77	1386.44	x_max
14	51	0.850	50	100.0	30.67	65.50	1190.59	354.02
15	57	0.950	60	109.1	37.48	70.53	1486.89	v_max
16	62	1.033	70	120.0	43.52	74.30	2088.65	91.94
17	67	1.117	80	120.0	49.85	77.68	1790.96	
18	72	1.200	90	133.3	56.45	80.68	2772.16	
19	76	1.267	100	150.0	61.91	82.81	4514.01	
20	80	1.333	110	133.3	67.49	84.71	2364.30	
21	85	1.417	120	133.3	74.64	86.75	2169.71	
22	89	1.483	130	150.0	80.47	88.13	3827.53	
23	93	1.550	140	150.0	86.39	89.29	3685.61	
24	97	1.617	150	150.0	92.37	90.23	3572.26	

Figure 6. Screenshot of the intial spreadsheet setup.

Note that for the first and last data points this *double-sided* estimate must be replaced by an appropriate *single-sided* estimate.

3. **Create individual cells for: M_0, k, x_0, SSE, and Δt.**

   Use the value of x_0 found earlier in the project. Start with $k = 0.5$ and $M_0 = 100$ as an initial guess.

4. **Choose either the constant-torque model or linear-torque model.**

   This choice affects the formula for $\widetilde{M}(x)$ that is used in the improved Euler Method. The constant-torque model uses (5), while the linear-torque model uses (6). It will be necessary to use a conditional logic function (typically called `IF`) to implement these formulas.

5. **Implement the improved Euler method shown in Figure 5 using additional rows.**

   Ten columns will be needed, as shown in Figure 7. It would be a good idea to use a step size Δt equal to the reciprocal of the frame rate for the camera. Reference the appropriate values of x_k and v_k to fill in Columns E and F, above. For example, in order to estimate $x(1.483)$ in Column E with $\Delta t = \frac{1}{60}$, the value of x_{89} should be referenced.

	A	B	C	D	E	F	G	H	I	J
26	n	t_n	x_n	v_n	m1	n1	x_temp	v_temp	m2	n2
27	0	0.00	0	0	0	100.000	0	1.6666667	1.6666667	99.167
28	1	0.02	0.0138889	1.6597222	1.6597222	99.164	0.0415509	3.3124517	3.3124517	98.325
29	2	0.03	0.0553237	3.3054596	3.3054596	98.322	0.1104147	4.9441578	4.9441578	97.477
30	3	0.05	0.1240705	4.9371193	4.9371193	97.475	0.2063558	6.5616947	6.5616947	96.624
31	4	0.07	0.2198939	6.5546111	6.5546111	96.622	0.3291375	8.1649749	8.1649749	95.767
32	5	0.08	0.3425571	8.1578474	8.1578474	95.764	0.4785213	9.7539131	9.7539131	94.904

Figure 7. Screenshot of the spreadsheet implementation of the improved Euler method.

6. **Add scatter plots of the observed and predicted values of $x(t)$ and $v(t)$.**

7. **Adjust the parameters M_0 and k to improve the accuracy of the model.**

   Make small adjustments to the parameters while watching the graphs and the sum of the squared error. Think about how each change will affect the graph before implementing it. If the spreadsheet is equipped with a solver, feel free to use the solver to optimize the values of M_0 and k so that the sum of the squared errors is minimized.

8. **Create a new sheet and repeat the preceding steps using the second model.**

   Only the formula for $\widetilde{M}$ will change, which affects the calculation of the slopes n_1 and n_2. The optimal values for M_0 and k are likely to change.

Questions:

1. How well do the optimized models fit the data? How do the parameters M_0 and k compare?
2. What is the maximum speed predicted by each model? It may be necessary to extend your numerical solution if the toy is still accelerating at the end of the observed data.
3. What is the maximum distance that the pull-back toy can travel under the power of the pull-back motor? As above, you will likely need to extend your numerical solution to answer this question.
4. How far can the pull-back toy actually roll? Make a few tests and compare the results with the mathematical models. Does this information suggest which model more accurately describes the motion of the pull-back toy? (The pull-back toy of Figure 1 can roll somewhere between 400 and 500 `cm` when fully wound.)

REFERENCES

[1] Wikipedia article: 'Pullback motor'. `https://en.wikipedia.org/wiki/Pullback_motor`. Accessed 10 June 2021.

[2] YouTube video: 'How does a Pull-Back Toy Car work?' `https://www.youtube.com/watch?v=QdvfiVebb_s`. Accessed 15 June 2021.