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STUDENT VERSION

Projectile Motion with Altitude-Dependent Gravity

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INTRODUCTION AND SCENARIO DESCRIPTION

In the most basic form of projectile motion, we assume that there is no air resistance, and that gravity is a constant g. Then, the ordinary differential equation (ODE) that describes the resultant vertical motion is:

$$y'' = g$$

where $y(t)$ is the height of the projectile at time t. Let us use the convention that $y > 0$ are heights above the surface and correspondingly $g < 0$. Let us also assume initial conditions:

$$y(0) = 0, \qquad y'(0) = v_0$$

where the object is launched from the surface $y = 0$, and that $v_0 > 0$ is a known upward initial velocity. Let us write y' as v, so that v is the vertical velocity. We then have:

$$v' = g$$

1. Integrate both sides of this equation with respect to dt, and apply one initial condition to solve for the unknown constant. Then change v back to y', integrate again, and apply the other initial condition.

To check your answer for part **1**, you should have gotten:

$$y(t) = \frac{1}{2}gt^2 + v_0 t,$$

which is a well-known result in physics. Among the many modeling assumptions that goes into this ODE is that gravity g is a constant. In reality, we know that gravity is a function of height above the surface. In particular, as per Newton's law of universal gravitation, it is proportional to the inverse-square of the distance r between the center of mass of the Earth (or whatever heavenly body we happen to be on), and the center of mass of our projectile.

Let us call the gravity our projectile experiences $\gamma = \gamma(r)$.

$$\gamma \propto \frac{1}{r^2}$$

Since we are calling the surface of the Earth $y = 0$, we can shift coordinates by arguing that $r = y + R$ where R is the radius of the Earth and $y > 0$. That gives us:

$$\gamma \propto \frac{1}{(y+R)^2}$$

and the constant of proportionality can be found since we know $\gamma = g$ when $y = 0$ (or equivalently, when $r = R$).

$$\gamma = \frac{gR^2}{(y+R)^2}$$

If we go back to our original ODE and replace g with γ, we have:

$$y'' = \frac{gR^2}{(y+R)^2}.$$

which is all well and good. But this ODE is very hard to solve, because it is nonlinear!

2. Why is this ODE nonlinear?

Let us linearize the ODE to make it easier to solve. The solution we get then will be an approximate solution, but we expect it to still be reasonable as long as $y \approx 0$. The benefit is that the linear version is much easier to solve.

3. Linearize the right-hand side of our ODE, which we will call $f(y)$. Our linearization is the first-order Taylor series, expanding around $y = 0$.

$$f(y) \approx f(0) + f'(0)y$$

within some interval of convergence around $y = 0$. Remember that y is our independent variable and g and R are constants.

4. Use this new $f(y)$ as our right hand side; this is the equation we'll be using for the rest of this exercise. Characterize this equation: What is its order? Is it linear or nonlinear? Homogeneous or non-homogeneous?

5. Use your knowledge of this type of ODE to find the general solution. Your answer should be in terms of the constants g and R.

Next, we can rewrite the answer to part **5** which will make the algebra a bit easier later on. We will make use of the hyperbolic trig functions "hyperbolic sine" (sinh) and "hyperbolic cosine" (cosh), which are defined as follows:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$
$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

Using these functions, we can rewrite something which looks like:

$$c_1 e^x + c_2 e^{-x}$$

Split each term into two halves:

$$\left(\frac{c_1}{2}e^x + \frac{c_1}{2}e^x\right) + \left(\frac{c_2}{2}e^{-x} + \frac{c_2}{2}e^{-x}\right)$$

Include some terms which add up to zero:

$$\left(\frac{c_1}{2}e^x + \frac{c_1}{2}e^x\right) + \left(\frac{c_2}{2}e^{-x} + \frac{c_2}{2}e^{-x}\right) + \underbrace{\left(\frac{c_2}{2}e^x - \frac{c_2}{2}e^x\right)}_{0} + \underbrace{\left(\frac{c_1}{2}e^{-x} - \frac{c_1}{2}e^{-x}\right)}_{0}$$

Factor:

$$= \frac{c_1 + c_2}{2}\left(e^x - e^{-x}\right) + \frac{c_1 - c_2}{2}\left(e^x + e^{-x}\right)$$

Rename:

$$\tilde{c_1} = c_1 + c_2, \qquad \tilde{c_2} = c_1 - c_2$$

And finally we have:

$$= \tilde{c_1}\cosh(x) + \tilde{c_2}\sinh(x)$$

6. Use the relationship above to rewrite your answer to part **5** in terms of sinh and cosh.

7. Apply the initial conditions $y(0) = 0$, $y'(0) = v_0$ to solve for the unknown constants in terms of v_0. Then rewrite the solution under these initial conditions. (Hint: some possibly useful facts about hyperbolic trig functions are given below.)

$$\sinh(0) = 0 \qquad \cosh(0) = 1$$
$$\frac{d}{dx}\sinh(x) = \cosh(x) \qquad \frac{d}{dx}\cosh(x) = \sinh(x)$$

Congratulations, you've solved the equations of motion! Now let's investigate how the graph looks visually and how it compares to the equations of motion when gravity was a constant. For starters, you can take the following values for the parameters:

$$R = 6.38 \times 10^6 \text{ m}$$
$$g = -9.8 \text{ m/s}^2$$
$$v_0 = 1000 \text{ m/s}$$

8a. Based on the graph, what is the highest altitude the projectile reaches? Compare this to our old equations of motion with constant gravity, given below. What was the highest altitude in that case?

$$y = \frac{1}{2}gt^2 + v_0 t$$

b. Answer the same questions as part **8a** but with $v_0 = 5000$ m/s instead. What do you notice?

c. Answer the same questions as **6a** but with $v_0 = 6000$ m/s instead. What do you notice here? The answer should be quite different than parts **8a** and **b**! Why does this happen?

d. (Bonus!) What is the condition on v_0 (in terms of g and R) where the behavior observed in **6c** will not occur?