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STUDENT VERSION

Differential Equations and Mathematical Models from Day One

William (Bill) Skerbitz
Math Department
Wayzata High School
Plymouth MN USA

INTRODUCTION AND SCENARIO DESCRIPTION

Differential Equations and Mathematical Modeling (Part I)

Unlimited Population Growth – In an unrestricted environment, it is reasonable to assume that at any given time, the rate of change in the size of a population P, denoted dP/dt or $P'(t)$, depends in some way on the size of the population at that time. More specifically, "the time rate of change of the population" (dP/dt) "is proportional to" (is equal to some constant multiple of) "the size of the population at that time t" (i.e., to $P(t)$ itself). In other words, the derivative (the rate of change) depends only on the population $P(t)$ (the dependent variable, which in turn depends on time t, the independent variable) and on k, a constant (or "parameter") of proportionality (sometimes called an "intrinsic growth rate coefficient") that is inherently connected to the population.

The assumptions above lead us to propose the following ***differential equation*** (an equation involving an unknown function and one or more of its derivatives) that models this situation:

$$\frac{d}{dt}(P(t)) = k \cdot P(t), \text{ or, after removing references to the independent variable, } \frac{dP}{dt} = kP.$$

Note how the verbal description translates into the equation:

$\frac{d}{dt}(P(t))$	The rate of change in the population at time t
$=$	is
$k\cdot$	directly proportional to
$P(t)$	the current population at time t

Here are some observations we can make about the proposed model

$$P'(t) = kP. \tag{1}$$

(i) Note that $P'(t) = 0$ when $P(t) = 0$. Mathematically, this means that the constant function $P(t) = 0$ is a ***solution*** to the differential equation (1) because on one hand, if $P(t) = 0$, then its derivative $P'(t)$ is 0 as well. On the other hand, if $P(t) = 0$ then $kP(t) = 0$ (regardless of the value of k). Therefore the LHS ("left-hand side") and the RHS ("right-hand side"), respectively, of (1) are equal when $P(t) = 0$. Constant functions $P(t) = C$ that are solutions to a differential equation $P'(t) = f(t, P)$ are called ***equilibrium solutions***. (The equilibrium solution is $P(t) = 0$ in this case.) Note that this claim also makes qualitative sense because if $P(t) = 0$, there is no population to change, i.e., it makes perfect sense that the corresponding rate of change in the population would be 0 when $P(t) = 0$.

(ii) If we assume $k > 0$ and $P_0 = P(t_0) > 0$ for some initial time t_0, then $P'(t) = kP$ will be positive as well. (In mathematical terms, $P(t)$ could be positive, negative, or zero, but in practical terms, it would only make sense for a population to be positive or zero.) Given $P > 0$ at time t_0, we would have $P'(t_0) = kP_0 > 0$, and so $P(t)$ must be increasing (because the derivative, $P'(t_0)$, is positive). Furthermore, with $k > 0$, the quantity $kP(t)$ will increase as $P(t)$ increases, implying that $P(t)$ is increasing at an increasing rate (i.e., the graph of $P(t)$ will be concave up.) Can you confirm this by finding $\frac{d^2P}{dt^2}$ implicitly in terms of P? Can you carry out a similar line of reasoning for $P_0 = 0$ or for $P_0 < 0$?

Note: It is customary to take constants encountered in mathematical modeling, such as the "k" in the above discussion, as being positive and explicitly introducing a minus sign into the model as an indication that the constant is meant to be of one specific sign or the other. However, be forewarned that this is not necessarily a universal practice, and therefore context continues to play an important role in reading and understanding mathematics. Incidentally, such constants are often referred to as ***parameters***.

The analysis in Tasks (i) and (ii) above demonstrate a ***qualitative analysis*** of the differential equation (1). We have yet to find an actual function rule that defines a solution to the differential equation, i.e., a function whose derivative behaves as described by the equation $dP/dt = kP$. We simply translated the words of the assumptions into mathematical statements in order to create a model and then used our knowledge about what a derivative indicates about a function to determine the behavior an actual solution of the model would exhibit. In most cases, qualitative analysis is a crucial part of understanding differential equations, and sometimes, it is the only analysis that is possible, particularly when modeling "real-world" phenomena.

Qualitative analysis gives us a relatively informed picture of how the model will behave in general.

On the other hand, ***analytic solutions*** (those determined algebraically via techniques from Calculus) are important if we wish to use the model to make accurate predictions. For example, if we wanted to estimate the population at a particular time, say at $t = 10$ or at $t = 100$, we would need to know what function $P(t)$ actually represents. If this is our intent, equation (1) from above and the ***initial condition*** $P(t_0) = P(0) = P_0$ together are called an ***initial-value problem*** (***IVP***), and for now, we will take a ***solution*** to the problem to be the function $P(t)$ that simultaneously satisfies both the differential equation and its accompanying initial condition.

As we progress through the course, you will learn techniques for determining analytic solutions for many types of equations. In this case, however, you should begin with an important fact from previous Calculus classes, namely, that

$$\frac{d}{dt}\left(e^{kt}\right) = ke^{kt}. \tag{2}$$

The significance of (2) cannot be overstated, as exponential functions form some of the most common solutions to differential equations we will encounter!

(iii) How are equations (1) and (2) related? Note that if we let $P(t) = e^{kt}$, then $P'(t) = ke^{kt}$, and since $e^{kt} = P(t)$, we see that $P'(t) = ke^{kt}$ becomes $P'(t) = kP$. This is exactly the differential equation we were analyzing in Tasks (i) and (ii) above! In addition, consider the following functions. Note how they all satisfy the equation $dP/dt = kP$:

(a) $P(t) = 3e^{kt} \implies P'(t) = 3ke^{kt} = k\left(3e^{kt}\right) = kP$

(b) $P(t) = -2e^{kt} \implies P'(t) = -2ke^{kt} = k\left(-2e^{kt}\right) = kP$

(c) $P(t) = \frac{1}{2}e^{kt} \implies P'(t) = \frac{1}{2}ke^{kt} = k\left(\frac{1}{2}e^{kt}\right) = kP$

(d) $P(t) = 0e^{kt} = 0 \implies P'(t) = 0 = k \cdot 0 = kP$

(e) $P(t) = Ce^{kt} \implies P'(t) = Cke^{kt} = k\left(Ce^{kt}\right) = kP$

(The process of differentiating a function to see that it satisfies a given differential equation is called ***verifying*** that the function is a solution to the equation.)

Task (iii)e shows us that any exponential function of the form $P(t) = Ce^{kt}$ could serve as a solution to the differential equation $dP/dt = kP$ for any given value of k. This solution represents an infinite family of solutions, one for each value of the ***parameter*** C, and is called the ***general solution*** of the differential equation. The function $P(t) = Ce^{kt}$ is therefore referred to as a "one-parameter family of solutions." How might we determine the value of the parameter C? This is where the other equation, the initial condition, comes into play.

The initial condition is represented by the equation $P(0) = P_0$. The P_0 in this case represents an unspecified constant (i.e., it represents the value of the function $P(t)$ at a particular value of t, namely $t = 0$ in this case). Since we know from Task (iii)e that one solution is given by $P(t) = Ce^{kt}$, we can determine the value of C by setting $t = 0$ and $P(t) = P(0) = P_0$ and solving for C, obtaining $C = P_0$. Then, substituting this value back into the solution from Task (iii)e, we have $P(t) = P_0 e^{kt}$. This solution, with the value of C determined, is called the ***particular solution*** to the initial value problem. Note that the general solution from Task (iii)e can, in this case, be used to determine the particular solution for any initial-value problem. (This is not the case for all differential equations.)

To summarize this crucial and key idea: An initial value problem of the form $dP/dt = kP$ with $P(0) = P_0$ has the functions $P(t) = Ce^{kt}$ and $P(t) = P_0 e^{kt}$ as its general and particular solutions, respectively.

Differential Equations and Mathematical Modeling (Part II)

A new and improved model – Note that the ***intrinsic growth rate*** k in our previous model $P' = kP$ remains constant. In practical terms, this would imply that regardless of how large the population is, the rate of growth remains the same. In particular, if we rewrite the equation as $k = P'/P$, and if P is counting the number of "critters" (and P' therefore has units of "critters per unit time") then we see that the growth rate k indicates how many critters per unit time are added to the population per critter in the population, and this quantity is by assumption a constant. Furthermore, we see that P'/P has units (critters/time)/critters = 1/time, so the expression "kt" in the exponent of the solution $P(t) = P_0 e^{kt}$, where t represents time, makes sense! (It is a good practice to always check the units in your equations.) It is the growth rate of the population, P'/P, having units of 1/(time), that interests us.

Since a constant growth rate for all time and environmental conditions does not seem likely, we might take into consideration limiting environmental factors such as space or resources. As a population increases, resources have to be shared between more and more members of the population. Under this new premise we realize that the growth rate is actually a function of the population itself, so even though we have the same three quantities as above (t, P, and k), instead of having a constant growth rate $k = P'/P$ from the differential equation $P' = kP$, we have a *variable* growth rate that depends on the population. Thus the growth rate k is itself a function of the population P, written as $K(P)$ instead of just the constant k. This would lead to a new differential equation $P' = K(P) \cdot P$ with its accompanying growth rate $P'/P = K(P)$, where the right-hand-side again indicates that the growth rate does not remain constant and instead depends on the population. We make some observations about this non-constant growth rate below.

- Trivially, if the population is zero, the rate of growth will also be 0. Therefore, when $P = 0$, $K(P) = 0$.

- If the population is small, the rate of growth behaves like the exponential model above, hence $K(P) \approx k$.

- If the population is near the maximum supportable by the environment (called the ***carrying capacity***) the rate of growth slows toward no growth (at the carrying capacity itself), hence $K(P) \approx 0$.
- Furthermore, if the population is too large for its environment to support, the population will begin to decrease in size, hence $K(P) < 0$.

In addition, we also introduce the parameter M to denote the aforementioned "carrying capacity," the population level at which the population becomes "too large" and its growth slows to a halt.

(iv) How can one determine the function $K(P)$? The behavior of this variable growth rate is such that (1) as the population increases, the growth rate decreases, so that if the population were at the carrying capacity M the growth would vanish, and (2) when the population is nowhere near the carrying capacity M, the growth would be similar to the constant growth rate experienced by a population undergoing exponential growth as discussed above. We are therefore assuming that the growth should be roughly k when P is small (say, relatively close to zero), and it should be zero when the population is at the carrying capacity M. This might suggest that the growth rate function $K(P)$ itself should pass through the two points $(0, k)$ and $(M, 0)$ and change continuously in between. What is the "simplest" continuous relationship between two points? Or, in somewhat non-mathematical terms, if the relationship is not constant, then what is the next simplest relationship between two variables? Using the population P as the independent variable and the growth rate K as the dependent variable, write the simplest equation you can that relates these two points on the graph of $K(P)$ versus P and solve the equation for $K(P)$. Your answer should involve k, K, P, and M. **Check with your instructor before you proceed.**

(v) Returning to the equation mentioned above, namely $P' = K(P) \cdot P$, replace $K(P)$ with the function found in Task (iv) and expand the expression so that it is in the form $P' = k \cdot (?)$, where the "?" will be some algebraic expression involving P and M.

(vi) Now, confirm that the behavior of P' from Task (v) matches the physical behavior we expect. For example, with $k > 0$, if the population is small (as in part (vi)b below), then the growth rate of the population should be positive, but relatively small, i.e., close to zero. Indeed, a P close to zero yet positive substituted into the equation from Task (v) would give us a small but positive growth rate. Similarly confirm each of the following for the differential equation from Task (v). The "should be..." part of the statement should be a verbal description of the growth behavior we would expect (increasing slowly, increasing rapidly, decreasing slowly, close to zero, leveling off, near zero but increasing, etc.).

(a) If $P = 0$, the size of the population should be __________, thus $P' =$ __________.

(b) If $P > 0$ but $P \approx 0$, the size of the population should be __________, thus $P' \approx$ __________.

(c) If $P < M$ but $P \approx M$, the size of the population should be ____________, thus $P' \approx$ ____________.

(d) If $P = M$, the size of the population should be ____________, thus $P' =$ ____________.

(e) If $P > M$ but $P \approx M$, the size of the population should be ____________, thus $P' \approx$ ____________.

(f) If P is MUCH larger than M, the size of the population should be ____________, thus P' should be ____________.

(g) In general, for $0 < P < M$, the size of the population should be ____________.

(h) In general, for $P > M$, the size of the population should be ____________.

(vii) Now that you have confirmed the behavior of the model, rewrite the differential equation again below, and **check with your instructor before you proceed.**

The differential equation obtained in Tasks (iv) through (vii) is called a ***logistic model*** with growth rate k and carrying capacity M. It is, like the exponential differential equation encountered earlier, a ***first-order*** differential equation because it only involves first derivatives of the dependent variable P. However, it is a ***non-linear*** equation because the right-hand side involves powers (other than one) of the dependent variable P (in other words, the right-hand side is not a linear function of P).

(viii) Whereas we could easily recall or even guess a solution to the equation $P' = kP$, we will have no such luck with the logistic model until we review how to solve ***separable*** differential equations and partial fraction decomposition. However, we can perform a qualitative analysis of the differential equation itself.

(a) In terms of P, what type of function results when you multiply out the right-hand side of the logistic differential equation (i.e., constant, linear, quadratic, cubic, etc.)? Use the set of dP/dt vs. P axes below to plot a rough sketch of (P, P'). (Note that this is P' vs. P, not P or P' vs. t.)

(b) What are the roots of this equation (make sure you are using the correct variables)? What do these roots correspond to as far as the growth of the population is concerned, and in contextual terms, why does this make sense? What are these solutions called (refer back to Task (i) if necessary)?

(ix) Now consider the graph (and its roots) you found in Task (viii) to answer the following:

(a) What is true about dP/dt for values of P less than the smallest root you found? Use this to complete the statement: "For any value of P less than __________, the value of P will (decrease / increase / neither)." (You will have to ignore the context here and just think in mathematical terms.)

(b) What is true about dP/dt for values of P between the roots you found? Give a corresponding statement about P similar to that in part (a).

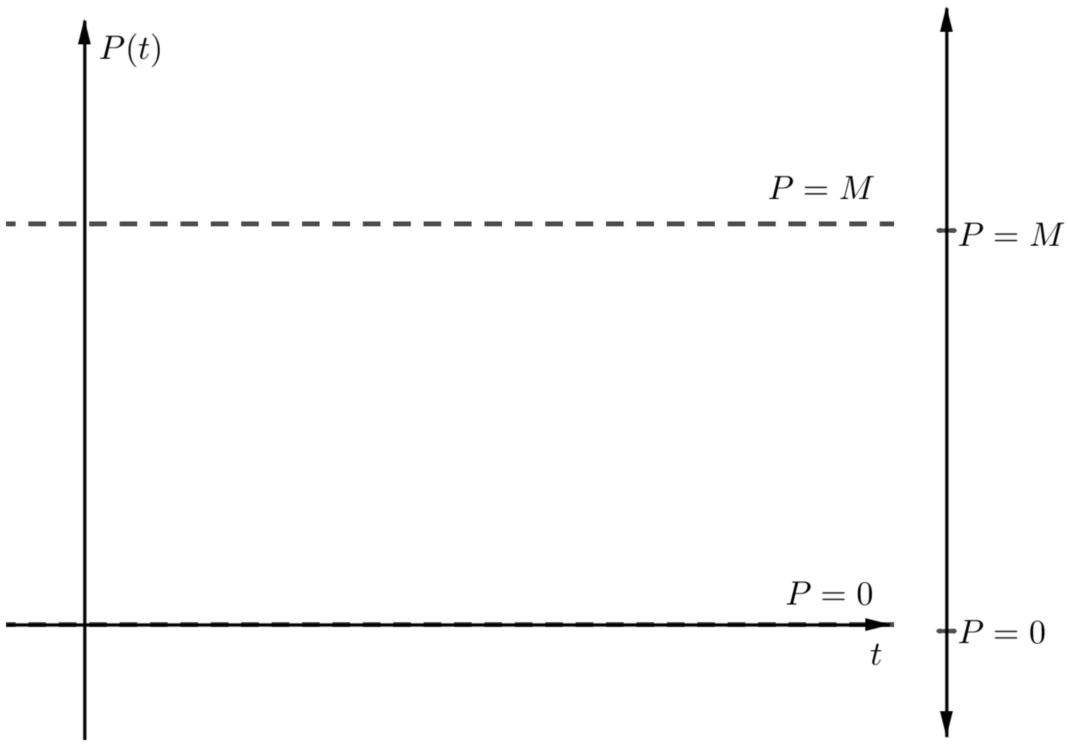
(c) What is true about dP/dt for values of P greater than the largest root you found? Give a corresponding statement about P similar to that in parts (a) and (b).

(d) Finally, note that the graph of the equation $P' = f(P)$ from part (a) of Task (viii) has a maximum – do you recall an easy way of finding the P-coordinate of this maximum without using Calculus? Do so. Since this is the value of P at which P' has a maximum, what does this say about the growth of the population?

(e) In terms of the shape of the curve, what graphical feature is present on the graph of P when P' has a local maximum as indicated in part (d)?

(x) What you should have found in Tasks (viii) and (ix) is that with very little information, we are able to come up with a relatively complete picture of the growth of the population P. Using all the information discovered in Tasks (viii) and (ix), sketch numerous solution curves on the (t, P) axes below for values of P that start below 0, at 0, between 0 and M, at M, and above M. Use arrows on the solution curves to show the end behavior (what happens as more and more time passes) of the population in each situation. (Pretend for the sake of mathematical completeness that a negative value of P is possible.)

(BONUS) – Once you have drawn the solutions as directed in Task (x) – can you imagine a way to "compress" all the relevant information about the behavior of $P(t)$ shown in those solutions onto the single vertical line shown to the right of the P vs. t graph? If you can, GREAT! That is a useful tool we'll discuss in the near future. HINT: Just three strategically placed arrowheads will do the trick.



Differential Equations and Mathematical Modeling (Part III)

Predator-Prey Systems – Rarely do species live in isolation. What if we wanted to model the interaction *between* species, as in the case where one species might be higher up the food chain than the other? Now we would have two quantities that depend on time and also affect each other. Suppose we are dealing with foxes (F) and mice (M). We make the following assumptions:

- Foxes are the only predators of mice, so if there are no foxes, the mice population grows uncontrolled at a rate proportional to itself.

- The foxes eat mice, and the rate at which the mice are eaten will be proportional to the rate at which foxes encounter mice.

- Mice are the only food source for foxes, so if there are no mice, the fox population will decline at a rate proportional to itself.

- The birth rate of foxes is proportional to the number of mice eaten, which in effect is proportional to the number of interactions between the mice and the foxes.

The assumptions lead us to introduce some new parameters, all of which we assume are positive:

Parameter	Description
α	Growth rate of mice (1^{st} assumption)
β	Constant of proportionality describing the rate at which mouse-fox interactions lead to a mouse being eaten, i.e., the detriment to the mouse population (2^{nd} assumption)
γ	Death rate of foxes (3^{rd} assumption)
δ	Constant of proportionality describing the rate at which fox-mouse interactions lead to an eaten mouse benefiting the fox population (4^{th} assumption)

(xi) Now we'll put these parameters to use in making our model. The model will involve the symbols $M(t)$, $F(t)$, dM/dt, dF/dt α, β, γ, and δ. Since we have two quantities changing with respect to time, we'll need two separate differential equations, one for each of dF/dt and dM/dt.

(a) Consider dM/dt. What would this be without foxes?

(b) Consider dF/dt. What would this be without mice? (RECALL – the parameters are positive.)

(c) The rate at which mice are eaten is proportional to how often they meet up with a fox, so we need a mathematical way to describe these encounters. Think of the simplest way to connect F and M to model these assumptions – if there are more mice or more foxes, this term should grow, but if there are no mice or no foxes, this term should be zero. (HINT – as an example - if there are 200 mice and 10 foxes, how many fox-and-mouse interactions are possible?)

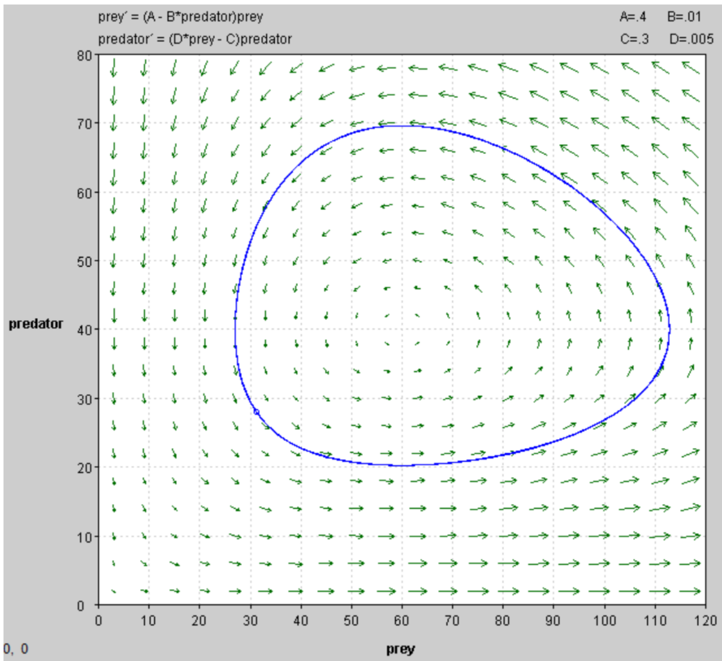
(d) **Check with your instructor** after parts (b) and (c). Then multiply the term from part (c) by the appropriate parameter from above and determine how to include it in the differential equation (Rationale: Not all of the fox and mouse interactions will result in a mouse being eaten, so some proportion of these interactions have an impact, and this is where the parameters come into play. Finally – is this interaction beneficial, or is it detrimental to the mouse population's growth? How about to the foxes? In other words, which equation is impacted, dF/dt or dM/dt, and how?)

(e) Now use the same thinking to determine the remaining differential equation. Write your result below, and **check with your instructor before proceeding**.

The result of your work in Task (xi) should be a pair of differential equations, both of which are first-order. This is called a ***first-order system*** of ordinary differential equations. This system is ***coupled*** because the rates at which F and M change depend on both F and M. Any solution to this system would require not one, but two functions that describe the populations of mice and foxes over time, and because the system is coupled, we need to solve these equations simultaneously, which is a very different process from solving systems of algebraic equations. We will spend more time on this later in the course once we develop some general theory and solution approaches!

Finally, let us bring our newfound (or perhaps "reawakened") skills in "qualitative analysis" to the study of a *system* of differential equations, but this time from a graphical perspective. Without having any idea whatsoever of what the algebraic "solutions" are or how they might be obtained, use your knowledge of what the derivative of a function indicates about the behavior of that function to make sense of the following graphs of a "predator-prey" system below. Using the questions accompanying the graphs below as a guide, work with your partner(s) to make sure each of you can explain to the other(s) what is being conveyed by each graph.

Graphical interpretations and analysis



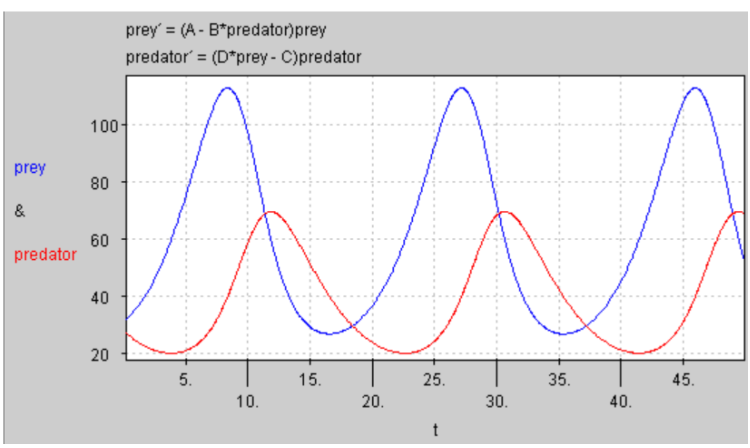
Example of a predator vs. prey "phase portrait." Pick a starting point on the blue curve, say the one marked at $(31, 28)$, and follow the arrows ... fill in the statements below:

Starting at _______ predators and _______ prey, the number of prey (inc. or dec.) while the number of predators (inc. or dec.).

Continue around the loop, describing how the predator and prey populations change along the way. Do you see how periodic behavior is established?

Can you explain in practical terms why the "system" would behave this way?

Can you describe the behavior of the system for the initial condition $(60, 40)$?



Individual plots of predators and prey over time, starting at initial point $(\text{prey},\text{pred}) = (31, 28)$

Try to understand how the blue graph at left tracks the "prey" population's growth and decline starting at prey $= 31$ from the phase portrait graph above.

Likewise, try to see how the red graph at left tracks the "predator" population's growth and decline starting at 28.